semicircle $S$ approaches $0$ as $R \to \infty$. For $r \to 0$ the integral over $C_2$ (clockwise) approaches the value

$$K = -\pi i \, \text{Res} \, f(z)$$

by Theorem 1. Together this shows that the principal value $P$ of the integral from $-\infty$ to $\infty$ plus $K$ equals $J$, hence $P = J - K = J + \pi i \, \text{Res}_{z=0} f(z)$. If $f(z)$ has several simple poles on the real axis, then $K$ will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

$$(14) \quad \text{pr. v. } \int_a^b f(x) \, dx = 2 \pi i \sum \text{Res} \, f(z) + \pi i \sum \text{Res} \, f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

**Example 4**

**Poles on the Real Axis**

Find the principal value

$$\text{pr. v. } \int \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}$$

**Solution.** Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex $z$, has simple poles at

$$z = 1, \quad z = 2, \quad z = i, \quad z = -i$$

the principal value (showing details):

$$\text{pr. v. } \int \frac{dx}{x^2 + 1} = 2 \pi i \left( \sum \text{Res} \, f(z) \right) + \pi i \sum \text{Res} \, f(z)$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v. } \int \frac{dx}{(x^2 + 3x + 2)(x^2 + 1)} = 2 \pi i \left( \frac{-1}{20} + \frac{1}{5} \right) = \frac{\pi}{10}$$

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.
Problem 5  Determine the radius of convergence of the Taylor series of the function
\[ h(z) = \frac{2}{1 + \cosh(z)} \]
centered at the origin. Explain your answer.

Problem 6  Consider the function
\[ f(z) = \frac{\ln(z)}{z^2 + a^2}, \quad a > 0, \]
in the upper half plane. For \( z = re^{i\theta}, 0 \leq \theta \leq \pi \) we define
\[ \ln(z) = \ln(r) + i\theta. \]
Let \( C \) be the following contour:

\[ \quad \]

\[ \quad \]

\[ \quad \]

\[ \quad \]

\[ \quad \]

\[ a) \] Determine \( \oint_C f(z) \, dz \) by residue calculations.

\[ b) \] Show that the integrals over the half circles \( C_R \) and \( C_\varepsilon \) approach zero as \( R \to \infty \) and \( \varepsilon \to 0. \)

\[ c) \] Compute
\[ \int_0^\infty \frac{\ln(x)}{x^2 + a^2} \, dx. \]