semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

by Theorem 1. Together this shows that the principal value P of the integral from $-\infty$ to ∞ plus K equals J; hence $P = J - K = J + \pi i \operatorname{Res}_{z=a} f(z)$. If $f(z)$ has several simple poles on the real axis, then K will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

$$\text{(14)} \qquad \text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

EXAMPLE 4 Poles on the Real Axis

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i \left(\frac{3-i}{20}\right) + \pi i \left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10}. \quad \blacksquare$$

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.

PROBLEM SET 16.4

1–4 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals and show the details of your work.

1. $\displaystyle\int_0^{\pi} \frac{2\,d\theta}{k - \cos\theta}$
2. $\displaystyle\int_0^{2\pi} \frac{1 + \sin\theta}{3 + \cos\theta}\,d\theta$
3. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$
4. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\,d\theta$

5–8 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your work.

5. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3}$
6. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1}\,dx$
7. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$
8. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{8 - x^3}\,dx$

9–14 IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

9. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$
10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$
11. $\displaystyle\int_{-\infty}^{\infty} \frac{x + 5}{x^3 - x}\,dx$
12. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\,dx$
13. **CAS EXPERIMENT. Simple Poles on the Real Axis.** Experiment with integrals $\int_{-\infty}^{\infty} f(x)\,dx$, $f(x) = [(x - a_1)(x - a_2)\cdots(x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.
14. **TEAM PROJECT. Comments on Real Integrals.**
    (a) Formula (10) follows from (9). Give the details.
    (b) Use of auxiliary results. Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a, a, a + ib, -a + ib$, letting $a \to \infty$, and using
    $$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$
    show that
    $$\int_0^{\infty} e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$
    (This integral is needed in heat conduction in Sec. 12.7.)
    (c) Inspection. Solve online Probs. 1 and 2 without calculation.

CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS

1. What is a Laurent series? Its principal part? Its use? Give simple examples.
2. What kind of singularities did we discuss? Give definitions and examples.
3. What is the residue? Its role in integration? Explain methods to obtain it.
4. Can the residue at a singularity be zero? At a simple pole? Give reason.
5. State the residue theorem and the idea of its proof from memory.
6. How did we evaluate real integrals by residue integration? How did we obtain the closed paths needed?
7. What are improper integrals? Their principal value? Why did they occur in this chapter?
8. What do you know about zeros of analytic functions? Give examples.
9. What is the extended complex plane? The Riemann sphere R? Sketch $z = 1 + i$ on R.
10. What is an entire function? Can it be analytic at infinity? Explain the definitions.

Problem 5 Determine the radius of convergence of the Taylor series of the function

$$h(z) = \frac{2}{1 + \cosh(z)}$$

centered at the origin. Explain your answer.

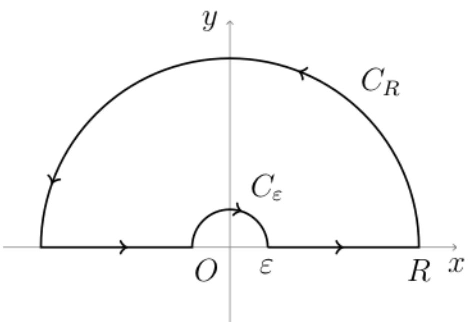
Problem 6 Consider the function

$$f(z) = \frac{\ln(z)}{z^2 + a^2}, \quad a > 0,$$

in the upper half plane. For $z = re^{i\theta}, 0 \leq \theta \leq \pi$ we define

$$\ln(z) = \ln(r) + i\theta.$$

Let C be the following contour:



a) Determine $\oint_C f(z)\,dz$ by residue calculations.

b) Show that the integrals over the half circles C_R and C_ε approach zero as $R \to \infty$ and $\varepsilon \to 0$.

c) Compute

$$\int_0^\infty \frac{\ln(x)\,dx}{x^2 + a^2}.$$