NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4120 Matematikk 4K**

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Other information:

- The problems 1a, 1b, 2, 3, 4, 5, 6, 7a, 7b, 7c have equal weight.

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Informasjon om trykking av eksamensoppgave	
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Date Signature

Problem 1

a) Find the function $f(t)$ that has the Laplace transform

$$F(s) = \mathcal{L}(f(t)) = \frac{11 - s}{s^2 - 2s - 3}.$$

b) Solve the initial value problem

$$\begin{cases} y''(t) = y'(t) + \delta(t-1) + 2u(t-1), \\ y(0) = 0, \; y'(0) = 1. \end{cases}$$

Problem 2 Expand

$$g(x) = \begin{cases} -\sin(x) & -\pi \le x \le 0, \\ \sin(x) & 0 \le x \le \pi, \end{cases}$$

in a Fourier series (with period 2π). Use the result to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

(Hint: Euler's formula.)

Problem 3 Determine the function $u(x, y)$ such that the function

$$f(x + iy) = u(x, y) + i(x^3 - 3y^2x - y)$$

is analytic. (Hint: Cauchy-Riemann.)

Problem 4 The temperature of an insulated thread satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \qquad -\infty < x < \infty,\; t > 0.$$

The initial temperature is

$$u(x, 0) = f(x) \qquad -\infty < x < \infty.$$

Find the Fourier transform

$$U(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} dx.$$

(Note that you need not to find $u(x, t)$.)

Problem 5

We know that the solution

$$u(x,t) = \sum_{n=1}^{\infty} (C_n \cos(nct) + D_n \sin(nct)) \sin(nx)\,,$$

of the wave equation describes the vibrations of an elastic string with fixed ends

$$u(0,t) = u(\pi,t) = 0\,, \qquad t > 0\,.$$

Determine the coefficients C_n and D_n so that the initial conditions

$$\begin{cases} u(x,0) = 5\sin(2x) \\ u_t(x,0) = \sin(x) + 3\sin(7x)\,, \end{cases}$$

are satisfied.

Problem 6

The Laurent series

$$\frac{1}{(z-1)(z^2+1)} = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$$

converges at the point $z = 3$. At which of the following points

1. $z = -1 - i$,
2. $z = 0$,
3. $z = 1 + i$,
4. $z = -3$,

does it also converge? (Justify your answer.)

Problem 7

a) Compute the integral

$$\oint_C \frac{e^z}{1+e^{4z}} dz \,,$$

where C is the path along the rectangle in the picture

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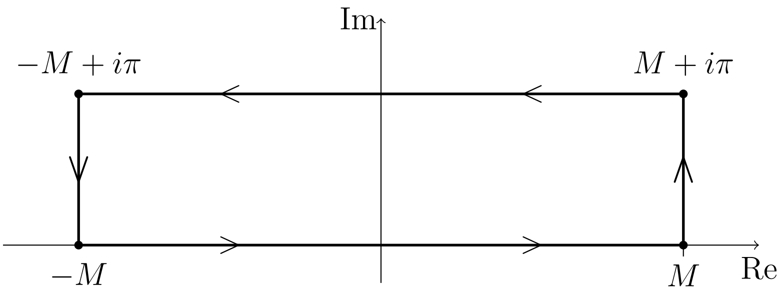

b) Show that the integrals in the vertical sides of C approach to zero when $M \to \infty$.

c) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{4x}} dx \,.$$

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at a and satisfies $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, moreover $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution** For functions defined on the real line:
  $f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad -\infty < x < \infty$;
  for functions defined only on the positive half-axis:
  $f * g\,(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0$.

Laplace transform

- $\mathcal{L}\{f\} = F(s) = \int_0^\infty f(t)e^{-st}dt$
- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$
- $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$
- $\mathcal{L}\left\{\int_0^t f(\tau)d\tau)\right\} = \frac{1}{s}\mathcal{L}\{f\}$
- $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
- $\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$, $c > 0$
- $\mathcal{L}\{tf(t)\} = -F'(s)$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- Periodic functions with period $2L$, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

- Parseval's identities $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$, $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$

- $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$
- $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$
- $\widehat{f'(x)} = iw\hat{f}(w)$
- $\widehat{f''(x)} = -w^2\hat{f}(w)$
- $f\widehat{(x-a)} = e^{-iaw}\hat{f}(w)$
- $\hat{f}(w-b) = \widehat{e^{ibx}f(x)}$
- $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$

$f(x)$	$\hat{f}(w)$
$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\begin{cases}1, & -b \le x \le b \\ 0, & \lvert x\rvert > b\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i\sin y)$,
  $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$

- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \; a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \; b_n = \frac{1}{2\pi i} \oint_C f(z)(z - z_0)^{n-1} dz$$

Some useful integrals

- $\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
- $\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
- $\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2 - a^2 x^2}{a^3} \cos ax + C$
- $\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2 - a^2 x^2}{a^3} \sin ax + C$
- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$
- $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$
- $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \; a > 0$