① a) $F(s) = \frac{11-s}{s^2-2s-3} = \frac{11-s}{(s-3)(s+1)} = \frac{2}{s-3} - \frac{3}{s+1} =$

$= 2\,\mathcal{L}(e^{3t}) - 3\,\mathcal{L}(e^{-t}) = \mathcal{L}(2e^{3t} - 3e^{-t})$

$\rightsquigarrow f(t) = 2e^{3t} - 3e^{-t}$

b) $\mathcal{L}(y'') = \mathcal{L}(y') + \mathcal{L}(\delta(t-1)) + \mathcal{L}(2u(t-1))$

$s^2Y - sy(0) - y'(0) = sY - y(0) + e^{-s} + 2\frac{e^{-s}}{s}$

$s^2Y - 1 = sY + e^{-s} + \frac{2e^{-s}}{s}$

$(s^2-s)Y = 1 + e^{-s} + 2\frac{e^{-s}}{s}$

$Y = \frac{1}{s^2-s} + \frac{e^{-s}}{s^2-s} + \frac{2e^{-s}}{(s^2-s)s}$

$Y = \frac{1}{s-1} - \frac{1}{s} + e^{-s}\left(\frac{3}{s-1} - \frac{3}{s} - \frac{2}{s^2}\right)$

$y(t) = e^t - 1 + u(t-1)\left(3e^{t-1} - 3 - 2(t-1)\right)$

$y(t) = e^t - 1 + u(t-1)\left(3e^{t-1} - 2t - 1\right)$

② $g(x) = \begin{cases} -\sin x & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$ even function, 2π-periodic.

$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}$, $b_n = 0 \quad \forall n$

obs $\sin x \cdot \cos nx = \frac{1}{2i}(e^{ix} - e^{-ix})\frac{1}{2}(e^{inx} + e^{-inx}) =$

$= \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right) dx$$

$$= \frac{1}{\pi}\left(\frac{\cos(n+1)x}{n+1} - \frac{\cos(n+1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} + \frac{1}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} + \frac{1}{n-1}\right)$$

$$= \begin{cases} -\frac{4}{\pi}\frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$g(x) = -\sum_{n=1}^{\infty} \frac{4}{\pi}\left(\frac{1}{(2n)^2-1}\right)\cos(2n)x + \frac{2}{\pi}$$

$$0 = g(0) = -\sum_{n=1}^{\infty} \frac{4}{\pi}\frac{1}{4n^2-1} + \frac{2}{\pi}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}}$$

(3) $u_x = v_y = -6yx - 1$

$u = \int u_x \, dx = -3yx^2 - x + C(y)$

$u_y = -3x^2 + C'(y) = -v_x = -3x^2 + 3y^2$

$\Rightarrow \quad C'(y) = 3y^2 \Rightarrow \quad C(y) = \int C'(y)\,dy = y^3 + K$

so $\quad u = y^3 - 3yx^2 - x + K.$

Then $\quad f(x+iy) = (y^3 - 3yx^2 - x + K) + i\,(x^3 - 3y^2x - y)$

(4) $\dfrac{\partial u}{\partial t} = \dfrac{1}{4}\dfrac{\partial^2 u}{\partial^2 x^2} - \dfrac{\partial u}{\partial x}$

$\downarrow$ Fourier transform w.r.t. x

$\hat{u}_t = -\dfrac{1}{4} w^2 \hat{u} + i w \hat{u}$

$\dfrac{d\hat{u}}{dt} = \hat{u}\left(\dfrac{-w^2 + 4iw}{4}\right)$

$\dfrac{1}{\hat{u}}\, d\hat{u} = \dfrac{-w^2 + 4iw}{4}\, dt$

$\ln \hat{u} = \dfrac{(-w^2 + 4iw)t}{4} + K(w)$

$\hat{u} = C(w)\, e^{\frac{(-w^2 + 4iw)t}{4}}$

and

$\hat{u}(w,0) = C(w)\, e^0 = \hat{f}(w)$

Then $\quad \hat{u}(w,t) = \hat{f}(w)\, e^{-\frac{w^2 + 4iw}{4} t}$

(5) $u(x,t) = \sum_{n=1}^{\infty} (C_n \cos(nct) + D_n \sin(nct)) \sin nx$

$u(x,0) = \sum_{n=1}^{\infty} C_n \sin nx = 5 \sin 2x \Rightarrow C_2 = 5 \ \& \ C_n = 0$
$\forall n \neq 2$

$u_t(x,t) = \sum_{n=1}^{\infty} (-n.c\, C_n \sin(nct) + nc\, D_n \cos(nct)) \sin nx$

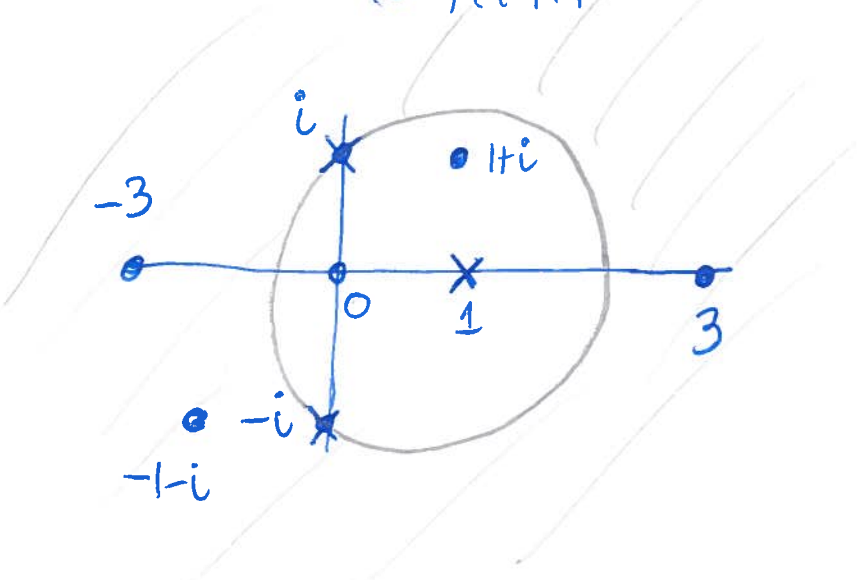
$u_t(x,0) = \sum_{n=1}^{\infty} nc \cdot D_n \sin nx = \sin x + 3 \sin 7x$

$1 = D_1 \cdot c \Rightarrow D_1 = 1/c$

$3 = D_7 \cdot 7c \Rightarrow D_7 = \frac{3}{7 \cdot c}$

$D_n = 0 \quad \forall n \neq 1, 7.$

(6) $f(z) = \frac{1}{(z-1)(z^2+1)} = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$

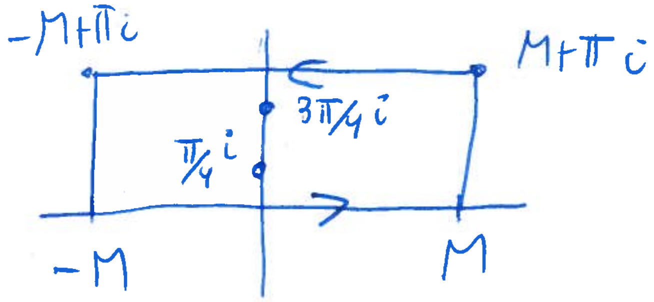


Converges in
$z = -3 \ \& \ z = -1-i$

⑦ a) $\displaystyle\oint_C \frac{e^z}{1+e^{4z}}\,dz$

$0 = 1 + e^{4z} \iff z = \frac{\pi}{4} + \frac{\pi}{2}k \quad k\in\mathbb{Z}$ are simple zeros

Then $z = \frac{\pi}{4} + \frac{\pi}{2}k \quad k\in\mathbb{Z}$ are simple poles



$$\operatorname*{Res}_{z=\pi/4\, i} \frac{e^z}{1+e^{4z}} = \frac{e^{\pi/4\, i}}{4e^{4\,\pi/4\, i}} = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

$$\operatorname*{Res}_{z=3\pi/4\, i} \frac{e^z}{1+e^{4z}} = \frac{e^{3\pi/4\, i}}{4e^{4(3\pi/4\, i)}} = \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

Then

$$\oint_C \frac{e^z}{1+e^{4z}}\,dz = 2\pi i\left(-\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i + \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i\right) = \frac{\pi\sqrt{2}}{2}$$

b) By the ML-inequality

$$\left|\int_{\text{right side}} \frac{e^z}{1+e^{4z}}\,dz\right| \le \frac{e^M}{e^{4M}-1}\cdot\pi \xrightarrow{M\to\infty} 0$$

$$|e^z| = |e^{x+iy}| = e^x|e^{iy}| = e^x = e^M$$

$$e^{4M} = |e^{z}|^4 = |e^{4z}| = |e^{4z}+1-1| \le |e^{4z}+1| - 1$$

$$\left| \int_{\substack{\text{left}\\ \text{side}}} \frac{e^{z}}{1+e^{4z}}\,dz \right| \le \frac{e^{-M}}{e^{-4M}-1}\,\pi \xrightarrow{M\to\infty} 0$$

$$|e^{z}| = e^{-M} \quad \& \quad e^{-4M}-1 \le |e^{4z}+1|$$

c) $$\frac{\pi\sqrt{2}}{2} = \overbrace{\int_{\substack{\text{right}\\ \text{side}}} \frac{e^{z}}{1+e^{4z}}\,dz}^{\to 0} + \overbrace{\int_{\substack{\text{left}\\ \text{side}}} \frac{e^{z}}{1+e^{4z}}\,dz}^{\to 0} +$$

$$+ \int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{4x}}\,dx - \int_{-\infty}^{\infty} \frac{e^{x+i\pi}}{1+e^{4(x+i\pi)}}$$

$$\frac{\pi\sqrt{2}}{2} = \int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{4x}}\,dx - \int_{-\infty}^{\infty} \frac{e^{x}\overbrace{e^{i\pi}}^{=-1}}{1+e^{4x}\underbrace{e^{4\pi i}}_{=1}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{4x}}\,dx + \int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{4x}}\,dx$$

$$= 2\int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{4x}}\,dx$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{4x}}\,dx = \frac{\pi}{2\sqrt{2}}}$$