TMA 4120 MATEM. 4K
14. XII . 2018

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(1a) $$F(s) = \frac{s^2+1}{s^2\underbrace{(s^2-4s+9)}_{(s-2)^2+5}}$$

$$= \frac{1/9}{s^2} + \frac{4/81}{s} + \frac{1}{81}\,\frac{-4s+88}{(s-2)^2+5}$$

$$= \frac{1/9}{s^2} + \frac{4/81}{s} - \frac{4}{81}\,\frac{(s-2)-20}{(s-2)^2+5}$$

$$f(t) = \frac{t}{9} + \frac{4}{81} - \frac{4}{81}e^{2t}\left\{\cos(\sqrt{5}\,t) - \frac{20}{\sqrt{5}}\sin(\sqrt{5}\,t)\right\}$$

(1b) $$\begin{cases} y''(t) - 4y'(t) + 9y(t) = t & (t>0) \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

$$(s^2-4s+9)\,Y(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$$

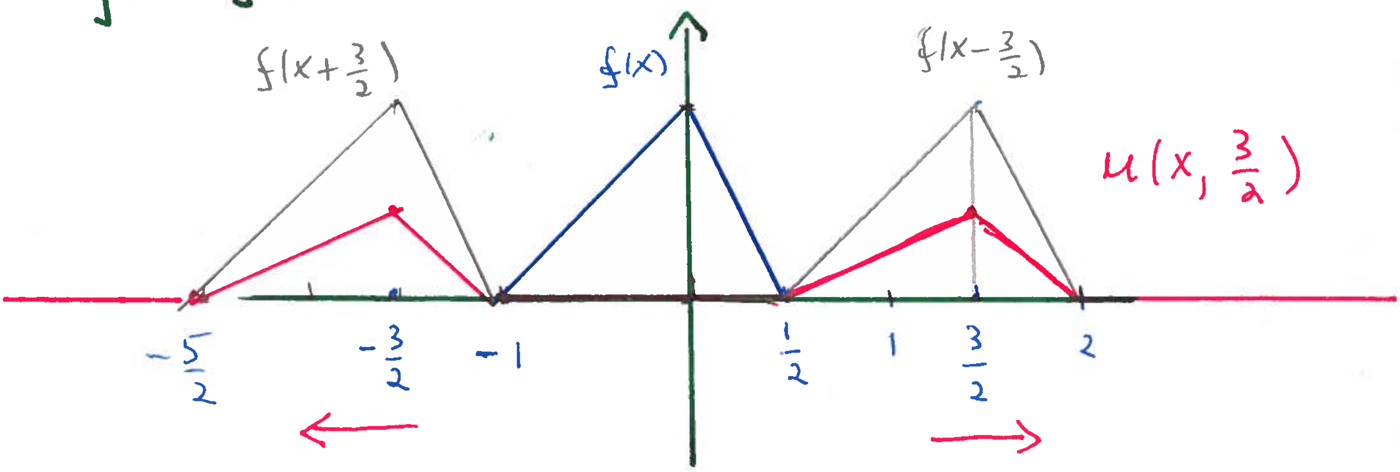
$$Y(s) = \frac{s^2+1}{s^2(s^2-4s+9)} = F(s) \text{ in (1a).}$$

$$y(t) = f(t) \text{ in (1a).}$$

(2) Use d'Alembert's formula

$$u(x,t) = \frac{f(x-t) + f(x+t)}{2} \qquad (c=1)$$

for $f(x) = u(x,0)$. We need $u(x, \frac{3}{2})$.



$u(x, \frac{3}{2})$ is the red wave, a broken line connecting the points

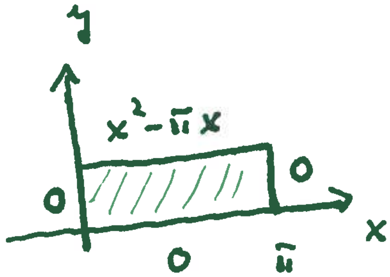
$x =$	$-\infty$,	$-5/2$	$-3/2$	-1	$1/2$	$3/2$	2	∞
$u =$	0	0	$1/2$	0	0	1	0	0

(3a) $$x^2 - \pi x = \sum_{n=1}^{\infty} b_n \sin(nx) \qquad 0 \le x \le \pi$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (x^2 - \pi x)\sin(nx)\,dx \qquad \text{[from Tables]}$$

$$= \begin{cases} 0, & n = 2, 4, 6, \dots \\ -\dfrac{8}{\pi n^3}, & n = 1, 3, 5, \dots \end{cases}$$

(3b) <u>Ansatz</u>: $u(x,y) = X(x)Y(y)$



$X(0) = 0, \quad X(\pi) = 0,$

$Y(0) = 0$

$$u_{xx} + u_{yy} - 20u = 0$$

$$X''Y + XY'' - 20XY = 0$$

constant of separation.

<u>Separation</u>:

$$\frac{X''}{X} = 20 - \frac{Y''}{Y} = \lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ Y'' = (20 - \lambda)Y \end{cases}$$

(I) $\boxed{X'' - \lambda X = 0}$, $X(0) = X(\pi) = 0$

- <u>$\lambda > 0$</u>. Only $X \equiv 0$ will do
- <u>$\lambda = 0$</u>. —— " —— —— " ——

- $\underline{\lambda < 0.}$

$$X(x) = \cancel{a\cos(\sqrt{-\lambda}\,x)} + b\sin(\sqrt{-\lambda}\,x)$$

$$X(0) = 0 \iff a = 0$$

$$X(\pi) = 0 \iff b\sin(\sqrt{-\lambda}\,\pi) = 0$$

$$\iff b = 0 \text{ or } -\lambda = n^2$$

We obtain

$$X_n(x) = b_n \sin(nx)$$

II $\boxed{Y'' = (20 + n^2)Y}$, $\quad Y(0) = 0$

$$Y(y) = a e^{\sqrt{20+n^2}\,y} + b e^{-\sqrt{20+n^2}\,y}$$

$$Y(0) = 0 \iff a + b = 0$$

We obtain

$$Y_n(x) = c_n \sinh(\sqrt{20+n^2}\,y)$$

SUPERPOSITION

New constant $= c_n b_n$

$$u(x,y) = \sum_{n=1}^{\infty} d_n \sinh(\sqrt{20+n^2}\,y)\sin(nx)$$

$$x^2 - \pi x \stackrel{?}{=} u(x,1) = \sum_{n=1}^{\infty} \underbrace{d_n \sinh(\sqrt{20+n^2})}_{\text{Numerical coefficient.}} \sin(nx)$$

This is a Fourier Sine Series and the coefficients are

$$b_n = d_n \sinh(\sqrt{n^2+20})$$

where the b_n were calculated already in 3a.

Answer:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sinh(\sqrt{n^2+20}\cdot y)}{\sinh(\sqrt{n^2+20})} \sin(nx)$$

$$= -\frac{8}{\pi} \sum_{\substack{n=1 \\ n=1,3,5,7,\dots}}^{\infty} \frac{1}{n^3} \frac{\sinh(\sqrt{}\cdot y)}{\sinh(\sqrt{})} \sin(nx)$$

④ $$\frac{1}{z^2+1} = \sum_{n=-\infty}^{\infty} c_n (z-i)^n, \qquad c_n = ?$$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$= \frac{1/2i}{z-i} - \frac{1}{2i(\underbrace{z-i}+2i)}$$

$$= \frac{1/2i}{z-i} + \frac{1}{4i^2\left(1+\frac{z-i}{2i}\right)} \qquad \text{(Geometric series)}$$

$$= \frac{1/2i}{z-i} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z-i}{2i}\right)^n .$$

This is the Laurent expansion. It converges $\Longleftrightarrow$

$$0 < \left|\frac{z-i}{2i}\right| < 1$$

The expansion converges $\Longleftrightarrow$ $|z-i| < 2$, $z \neq i$.

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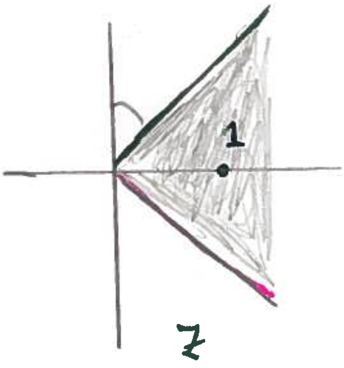
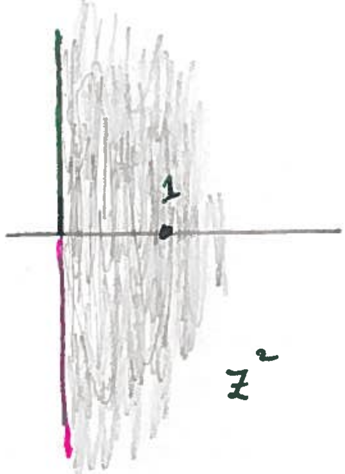
⑤ Answer $w = 2iz^2$



$\xrightarrow{z^2}$



The square z^2 opens the angle to $180°$. $2i$ rotates by $90°$ and changes the scale.

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$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{(x^2+1)(x^2+4)}\, dx$$

$$F(z) = \frac{e^{-i\omega z}}{(z^2+1)(z^2+4)}$$ has simple poles at $z = \pm i$, $z = \pm 2i$. The residues are

$$\operatorname*{Res}_{z=i} \{F(z)\} = \lim_{z \to i} (z-i)F(z) = \frac{e^{+\omega}}{2i \cdot 3}$$

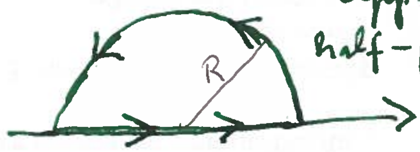
$$\operatorname*{Res}_{z=-i} \{F(z)\} = \lim_{z \to -i} (z+i)F(z) = \frac{e^{-\omega}}{-2i \cdot 3}$$

$$\operatorname*{Res}_{z=2i} \{F(z)\} = \frac{e^{+2\omega}}{-12i}$$

$$\operatorname*{Res}_{z=-2i} \{F(z)\} = \frac{e^{-2\omega}}{12i}$$

NOTICE $\left| e^{-i\omega(x+iy)} \right| = e^{\omega y}$. We must keep $\omega y < 0$ as $y \to +\infty$ or $-\infty$.

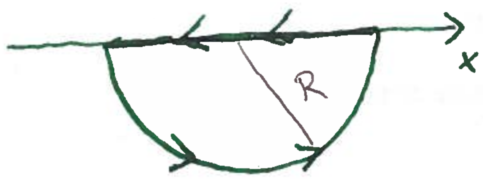
Hence the contour is like



Upper half-plane.

$\omega < 0$

$R \to \infty$



$\omega > 0$

$R \to \infty$

Lower half-plane

$\underline{\omega < 0}$ Only the residues in the upper half-plane count.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \left\{ \frac{e^{\omega}}{2i \cdot 3} - \frac{e^{2\omega}}{12 i} \right\}$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{3} \left\{ e^{\omega} - \frac{e^{2\omega}}{2} \right\}. \qquad (\omega < 0)$$

$\underline{\omega > 0}$ Only the residues in the lower half-plane count. The formula needs a minus sign in front to reverse the clockwise direction:

$$\hat{f}(\omega) = - \frac{1}{\sqrt{2\pi}} 2\pi i \left\{ \frac{e^{-\omega}}{-2i \cdot 3} + \frac{e^{-2\omega}}{12 i} \right\}$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{3} \left\{ e^{-\omega} - \frac{1}{2} e^{-2\omega} \right\} \qquad (\omega > 0)$$

Answer:

$$\hat{f}(\omega) = \frac{1}{3} \sqrt{\frac{\pi}{2}} \left\{ e^{-|\omega|} - \frac{1}{2} e^{-2|\omega|} \right\}$$

when $-\infty < \omega < \infty$.

REMARK: Using

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3} \left(\frac{1}{x^2+1} - \frac{1}{x^2+4} \right)$$

you can get $\hat{f}(\omega)$ from the table of transforms.