Shannon theorem

We say that signal $f(t)$ is bandlimited to $(-L, L)$

if it has representation

$$f(t) = \int_{-L}^{L} \hat{f}(\omega)\, e^{i\omega t}\, dt \qquad (1)$$

Shannon theorem gives precise reconstruction of such signal from

it samples at the points $\left\{\frac{n\pi}{L}\right\}_{n=-\infty}^{\infty}$.

Here how this works:

1) Write the Fourier series for the function $\hat{f}(\omega)$ on $[-L, L]$

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}\omega}$$

2) Substitute this series in (1):

$$f(t) = \sum_{-\infty}^{\infty} c_n \int_{-L}^{L} e^{i\omega t} e^{i\frac{n\pi}{L}\omega}\, d\omega$$

and evaluate the integrals in the right hand side:

$$\int_{-L}^{L} e^{i\omega t + i\frac{n\pi}{L}\omega}\, d\omega = 2\,\frac{\sin(Lt + n\pi)}{t + \frac{n\pi}{L}}$$

So $$f(t) = 2\sum_{n=-\infty}^{\infty} c_n \frac{\sin(Lt + n\pi)}{t + \frac{n\pi}{L}} \qquad (2)$$

3) Expression for the coefficients of the Fourier series:

$$c_n = \frac{1}{2L} \int_{-L}^{L} \hat{f}(\omega)\, e^{-i \frac{n\pi}{L}\omega}\, d\omega$$

Compare this with (1) gives $c_n = \frac{1}{2L} f\left(-\frac{n\pi}{L}\right)$

We substitute this in (2)

$$f(t) = \sum_n f\left(-\frac{n\pi}{L}\right) \frac{\sin(Lt + n\pi)}{Lt + n\pi}$$

This is the famous Shannon formula!

4) Remarks:

1. Distance between sampling points is $\frac{\pi}{L}$, it is called Nyquist rate, bigger L more often should we sample.

2. In real applications people use oversampling i.e. sample more often in order to provide better stability.