TMA4220 Calculus 4K, Practice problems, Fall 2018

Complex analysis, Residue calculus.

Problem 1. Let $f(z) = z(z + \bar{z})$.
a) Write $f(z)$ in the form $u(x, y)+iv(x, y)$ where $z = x+iy$ and u and v are real-valued functions.
b) Find all point on the complex plane where f is differentiable and compute f' at those points.
c) For each of the functions $u(x, y)$ and $v(x, y)$ determine if it is harmonic (on the whole complex plane) or not. Justify your answer.

Problem 2. Let $g(z) = \overline{f(\bar{z})}$.
a) If $f(z) = e^{(1+i)z}$, find $g(z)$.
b) Suppose that $f(x+iy)) = u(x, y)+iv(x, y)$, where u and v are real-valued. Find the real and imaginary parts of $g(x + iy)$. Show that if $f(z)$ is analytic at z_0 then $g(z)$ is analytic at $\bar{z}_0$.

Problem 3. a) Find all complex numbers z such that $e^z = -1$.
b) Show that $|e^z| \leq e^{|z|}$ for any complex number z. For which z there is equality?

Problem 4. Compute the integral $\int_C f(z)dz$, where $f(z) = \frac{z-1}{z+1}$ and
(i) C is the circle centered at -1 of radius 2.
(ii) C is the upper semicircle centered at -1 and of radius 2 (Hint: use parametrization).

Problem 5. Let $f(z) = \frac{z+2}{z^3-1}$ and C_R be the circle $\{|z| = R\}$.
a) Find all singular points and residues of the function $f(z)$. Compute the integral $\int_{C_2} f(z)dz$ using residues.
b) Use the LM-inequality to show that

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{2\pi R(R + 2)}{R^3 - 1}.$$

Let $R \to \infty$ and show that $\int_{C_2} f(z)dz = 0$.

Problem 6. Show that $\oint_C \bar{z}dz$ is equal to the area of the domain bounded by the curve C. (Hint: examine the proof of the Cauchy's integral formula and apply the Green's theorem to $\oint_C \bar{z}dz$.)

Problem 7. Let $f(z) = \frac{z}{z^2+1}$.
a) Find the Taylor series of $f(z) = \frac{z}{z^2+1}$ at $z_0 = 0$. What is the radius of convergence of this series?
b) Find all Laurent series of f centered at $z_1 = 1$.

Problem 8. Let $f(z) = \frac{z}{1-e^z}$.
a) Determine all singular points of f and classify them (removable singularities, poles, essential singularities).
b) Let $C = \{|z - 1| = 8\}$. Compute $\oint_C dz$

Problem 9. Let C be a simple closed curve on the complex plane and

$$g(a) = \oint_C \frac{z^3 + z^2 - 3}{(z - a)^3} dz.$$

Show that $g(z) = 0$ when a is outside C and $g(a) = 4\pi i(3a + 1)$ when a is inside C.

Problem 10. Evaluate the following integrals using residues:

$$a) \int_0^\pi \frac{d\theta}{(2 + \cos\theta)^2}, \quad b) \int_{-\infty}^\infty \frac{x^4}{x^6 + 4} dx, \quad c) \int_0^\infty \frac{x \sin x}{x^4 + 1} dx.$$

Problems for exams

Problem 1a. Find the values of the following functions $e^z, \cos z, \operatorname{Ln} z, i^z$ at $z_0 = i$.

Problem 2a. Let $f(z) = \frac{\cos z}{z^2+1}$.
a) Find all zeros and singular points of $f(z)$, classify the singularities.
b) Compute $\oint_C f(z)dz$, where C is the circle of radius 3 centered at $z_0 = 1$.
c) Let $g(w) = f(1/w)$ show that $g(w)$ has essential singularity at zero. (Hint: consider the values of g on a set $0 < |w| < r$.)

Problem 3a. Evaluate the integral $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$

Problem 1b. Let $u(x, y) = e^{2x} \cos by$.
a) Determine b such that $u(x, y)$ is harmonic.
b) Find $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function in the whole complex plane. Justify your answer.

Problem 2b. Consider the Taylor series at the origin

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

a) Use term-wise differentiation to find the Taylor series centered at $z_0 = 0$ of the function $f(z) = \frac{1}{(1-z)^3}$. Find the radius of convergence of this series.
b) Write down the other Laurent series of the function $f(z)$ with center $z_0 = 0$ and indicate where it converges.

Problem 3b. Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2}$.

Problem 1c. Which of the following functions are analytic at $z_0 = 0$ (Justify your answer):

$$(i)\ \operatorname{Re}(z), \quad (ii)\ |z|^2, \quad (iii)\ \tan z, \quad (iv)\ \sin\frac{1}{z}.$$

Problem 2c. Consider the series $\sum_{n=0}^{\infty} \frac{3^n}{2n+1} z^{2n+1}$.
a) Find the radius of convergence of this series.
b) Let $f(z)$ be the sum of the series, write down the series expansion of $f'(z)$ and find $f'(z)$.
c) Show that $f(z) = \operatorname{Ln}\sqrt{1 - 3z^2}$.

Problem 3c. Evaluate the integral $\int_0^{\infty} \frac{\cos 3x}{x^2+1} dx$.