TMA4120 Calculus 4K, Midterm Practice problems, Fall 2018

Laplace transform, Fourier series and Fourier transform, PDEs.

Problem 1. Sketch the graph of the function $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$ and find the Laplace transform of g.

Problem 2. Solve the initial value problem $y'' - 2y' + 2y = \delta(t-1) + e^{-t}$, $y(0) = 1,\ y'(0) = 0$, using the Laplace transform.

Problem 3. Consider the linear system of differential equations $x' = x + y, \quad y' = 4x + y$ with initial conditions $x(0) = 0$ and $y(0) = 2$ and let $X(s) = \mathcal{L}\{x\}(s)$ and $Y(s) = \mathcal{L}\{y\}(s)$ be the Laplace transforms of the functions $x(t)$ and $y(t)$, respectively.
a) Find $X(s)$ and $Y(s)$. b) Determine $x(t)$ and $y(t)$.

Problem 4. Use the Laplace transform to solve the equation $y'(t)+\int_0^t e^u y(t-u)du = t+y(t),\ t > 0$ with initial condition $y(0) = 1$.

Problem 5. Let $f(x) = x^2$ when $0 < x < 1$.
a) Find the coefficients of the sine Fourier series for f, $S_f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$. Sketch the graph of S_f on $[0, 3]$ and determine the values $S_f(0), S_f(1/2), S_f(1)$.
b) Find the coefficients of the corresponding cosine Fourier series of f, $C_f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$.and sketch the graph of the C_f on $[0, 3]$.
c) If you want approximate the function f by a partial sum F of one of the series from a) and b) such that the square error $\int_0^1 (f - F)^2 dx$ is less than 10^{-5}, which series would you use and why? How many terms would you need (a rough approximation)? Why do the coefficients of sine and cosine series behave so differently?

Problem 6. Let $f(x) = x^2(\pi - x)$ when $0 < x < \pi$, we extend f to a π-periodic function.
a) Find the complex Fourier series of f and convert this series to real form.
b) Find a particular solution of the differential equation $y'' + 3y' + 2y = f(x)$.

Problem 7. a) Let $\hat{f}(w)$ be the Fourier transform of some function $f(x)$. Compute the Fourier transform of $f(x - a)$ (show your work).
b) Find the Fourier transform of e^{-x^2+2px} (use your computation from a) and the formula $\mathcal{F}(e^{-kx^2}) = (2k)^{-1/2} e^{-w^2/(4k)}$).

Problem 8. Let $f(x) = 1$ when $-1 < x < 1$ and $f(x) = 0$ when $|x| > 1$.
a) Compute $\hat{f}(w)$.
b) Compute $h(x) = (f * f)(x) = \int_{-\infty}^{\infty} f(y)f(x - y)dy$.
c) Find $\hat{h}$ by the definition and then by the convolution theorem.

Problem 9. Consider the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi,\ t > 0$ with boundary condition $u(0, t) = u(\pi, t) = 0$ and initial data $u(x, 0) = 0$, $u_t(x, 0) = \sin x(1 + \cos x)$.
a) Find all functions of the form $u(x, t) = F(x)G(t)$ that solve the equation and satisfy the boundary conditions.
b) Solve the equation with boundary and initial conditions.

Problem 10. Solve the heat equation on the infinite string, $u_t = c^2 u_{xx}$ for $-\infty < x < \infty$ and $t > 0$ if the initial temperature is given by $u(x, 0) = xe^{-x^2}$. (Hint: you may use that $\mathcal{F}(xe^{-ax^2}) = -\frac{iw}{2a\sqrt{2a}} e^{-w^2/(4a)}$.)

Problem 1a. Let $f(t) = \begin{cases} t, \ 0 < t < \pi \\ \pi, \ t \geq \pi \end{cases}$.

a) Compute the Laplace transform of f.
b) Solve the initial value problem $y'' + 4y = f(t), \ t > 0, \ y(0) = 1, \ y'(0) = 0$.

Problem 2a. Consider the Laplace equation $u_{xx} + u_{yy} = 0$ for $0 < x < \pi$ and $0 < y < 2\pi$ with mixed boundary condition on the vertical sides: $u(0, y) = 0, \ u_x(\pi, y) = 0$
a) Find all solutions of the form $u(x, y) = F(x)G(y)$.
b) Solve the equation with given boundary conditions on the vertical sides and the following condition on the horizontal sides $u(x, 0) = u(x, 2\pi) = \sin \frac{3\pi x}{2} + 4 \sin \frac{7\pi x}{2} - 5 \sin \frac{11\pi x}{2}$.

Problem 3a. Find the inverse Fourier transform of the function $\frac{1}{(1+iw)^2}$. (Hint: you may use the convolution theorem and the formula $\mathcal{F}(e^{-x}u(x)) = \frac{1}{\sqrt{2\pi}(1+iw)}$, where $u(x)$ is the Heaviside function, or you may apply the residue calculus.)

Problem 1b. a) Find the inverse Laplace transform of

$$F(s) = \frac{s(s+2)}{s^3 + s^2 + s + 1}.$$

(Hint: $s^3 + s^2 + s + 1 = (s^2 + 1)(s + 1)$.)
b) Solve the integral equation $f(t) = \cos t + e^{-2t} \int_0^t f(\tau) e^{2\tau} d\tau$.

Problem 2b. Let $f(x)$ be the 2-periodic function such that $f(x) = 1 - |x|$ for $|x| < 1$.
a) Find the Fourier series of $f(x)$.
b) Find a particular solution of the differential equation $y'' + 9y = f(x)$.

Problem 3b. Compute the Fourier transform of the function $f(x) = \begin{cases} e^{-|x|} - e^{-1}, \ |x| < 1 \\ 0, \quad |x| \geq 1 \end{cases}$.

Problem 1c. Solve the initial value problem $y'' + 4y' + 4y = u(t - 1), \ y(0) = 0, \ y'(0) = 1$.

Problem 2c. Show that if $\hat{f}(w)$ is the Fourier transform of a function $f(x)$ then the Fourier transform of the function $f(x) \sin bx$ is equal to $\frac{i}{2}(\hat{f}(w + b) - \hat{f}(w - b))$. Compute the Fourier transform of the function $g(x) = \begin{cases} \sin 2x, \ |x| \leq 1 \\ 0, \ |x| > 1 \end{cases}$.

Problem 3c. Consider the following partial differential equation $u_{tt} + 2u_t + u = c^2 u_{xx}$ for $0 < x < L, \ t > 0$, with boundary condition $u(0, t) = u(L, t) = 0$.
a) Find all solutions of the form $u(x, t) = F(x)G(t)$.
b) Find the sine Fourier series $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ of the function $f(x) = x(L - x)$ for $0 < x < L$.
c) Solve the equation with the given boundary condition and the initial conditions $u(x, 0) = f(x), \ u_t(x, 0) = 0$