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Løysingsforslag – Øving 13

16.3.4

$$\begin{aligned} e^{\frac{1}{1-z}} &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{1-z}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(1-z)^n} \\ &= 1 + \frac{1}{1-z} + \frac{1}{2(1-z)^2} + \cdots \end{aligned}$$

Funksjonen har ein essensiell pol i $z = 1$. Residyen til f i $z = 1$ er koeffisienten til leddet $\frac{1}{z-1}$. Dvs.

$$\operatorname*{Res}_{z=1} f(z) = -1.$$

16.3.7 $\tan(2\pi z)$ is analytic inside C except at $z = 1/4$. The residue at the singular point is given by (4) on page 721, with $p(z) = \sin(2\pi z)$ and $q(z) = \cos(2\pi z)$:

$$\begin{aligned} \operatorname*{Res}_{z=1/4} \tan(2\pi z) &= \frac{\sin(2\pi/4)}{-2\pi \sin(2\pi/4)} \\ &= -\frac{1}{2\pi} \end{aligned}$$

(Since $q'(1/4) \neq 0$, the pole is simple, so we are allowed to use (4).) We calculate the integral by the residue theorem.

$$\begin{aligned} \oint_C \tan(2\pi z)\, dz &= 2\pi i \operatorname*{Res}_{z=1/4} \tan(2\pi z) \\ &= -i \end{aligned}$$

16.4.3 Set $z = e^{i\theta} \implies \cos\theta = (z + z^{-1})/2$, $\sin\theta = (z - z^{-1})/(2i)$, $d\theta = dz/(iz)$, slik at

$$\begin{aligned} \int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta} \mathrm{d}\theta &= -\frac{1}{4} \oint_C \frac{\left(z - z^{-1}\right)^2}{5 - 2\left(z + z^{-1}\right)} \frac{\mathrm{d}z}{iz}, \quad C : |z| = 1 \\ &= \frac{-1}{4i} \oint \frac{z^2 - 2 + z^{-2}}{5z - 2\left(z^2 + 1\right)} \mathrm{d}z \\ &= \frac{i}{4} \oint_C \frac{z^4 - 2z^2 + 1}{z^2\left(-2z^2 + 5z - 2\right)} \mathrm{d}z \\ &= \frac{-i}{8} \oint_C \frac{z^4 - 2z^2 + 1}{z^2(z-2)\left(z - \frac{1}{2}\right)} \mathrm{d}z \end{aligned}$$

Integranden $f(z)$ har ein andreordens pol i $z = 0$, ein førsteordens pol i $z = 2$ og ein førsteordens pol i $z = 1/2$. Av desse ligg $z = 0$ og $z = 1/2$ innanfor einingssirkelen. Integralet blir dermed

$$\begin{aligned}
\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\mathrm{d}\theta &= \left(\frac{-i}{8}\right) 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1/2} f(z)\right) \\
&= \frac{\pi}{4}\left(\lim_{z\to 0}\left(\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^4-2z^2+1}{z^2-\frac{5}{2}z+1}\right)\right) + \lim_{z\to 1/2}\left(\frac{z^4-2z^2+1}{z^2(z-2)}\right)\right) \\
&= \frac{\pi}{4}\left(\lim_{z\to 0}\left(\frac{\left(4z^3-4z\right)\left(z^2-\frac{5}{2}z+1\right)-\left(z^4-2z^2+1\right)\left(2z-\frac{5}{2}\right)}{\left(z^2-\frac{5}{2}z+1\right)^2}\right)\right. \\
&\quad \left. + \frac{2^{-4}-2\cdot 2^{-1}+1}{2^{-2}\left(2^{-1}-2\right)}\right) \\
&= \frac{\pi}{4}\left(\frac{5}{2}-\frac{3}{2}\right) \\
&= \frac{\pi}{4}
\end{aligned}$$

16.4.5 Funksjonen $f(z) = \frac{1}{(1+z^2)^3}$ har ein trippel pol i $z = i$ i det øverste halvplanet. Residyen blir

$$\begin{aligned}
\operatorname{Res} f(z) &= \frac{1}{2!}\lim_{z\to i}\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(z-i)^3\frac{1}{\left(1+z^2\right)^3}\right]\right\} \\
&= \frac{1}{2!}\lim_{z\to i}\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(z-i)^3\frac{1}{((z+i)(z-i))^3}\right]\right\} \\
&= \frac{1}{2}\lim_{z\to i}\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{1}{(z+i)^3}\right]\right\} \\
&= \frac{1}{2}\lim_{z\to i}\left\{\frac{(-3)(-4)}{(z+i)^5}\right\} \\
&= \frac{12}{2}\frac{1}{2^5 i} \\
&= \frac{3}{16i}.
\end{aligned}$$

Integralet blir dermed

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\left(1+x^2\right)^3} &= 2\pi i\operatorname*{Res}_{z=i} f(z) \\
&= 2\pi i\left(\frac{3}{16i}\right) \\
&= \frac{3\pi}{8}
\end{aligned}$$

16.4.12 Let $f(z) = \frac{z^2}{z^4-1}$. We have $z^4-1 = (z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i)$, which makes the poles of f and the corresponding residues straightforward to find.

$$
\begin{aligned}
\operatorname*{Res}_{z=1} f(z) &= \left[\frac{z^2}{(z+1)(z-i)(z+i)}\right]_{z=1} \\
&= \frac{1^2}{(1+1)(1-i)(1+i)} \\
&= \frac{1}{4} \\
\operatorname*{Res}_{z=-1} f(z) &= \frac{(-1)^2}{(-1-1)(-1-i)(-1+i)} \\
&= -\frac{1}{4} \\
\operatorname*{Res}_{z=i} f(z) &= \frac{i^2}{(i-1)(i+1)(i+i)} \\
&= -\frac{i}{4} \\
\operatorname*{Res}_{z=-i} f(z) &= \frac{(-i)^2}{(-i-1)(-i+1)(-i-i)} \\
&= \frac{i}{4}
\end{aligned}
$$

By (14), the Cauchy principal value is

$$
\begin{aligned}
2\pi i \operatorname*{Res}_{z=i} f(z) + \pi i \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z)\right) &= 2\pi i \frac{-i}{4} \\
&= \frac{\pi}{2}
\end{aligned}
$$

a Let C be the contour described in the problem text. We have

$$\int_C \frac{e^{iz}}{z \pm ia}\, dz = \int_{-R}^{R} \frac{e^{ix}}{x \pm ia}\, dx + \int_{|z|=R, y>0} \frac{e^{iz}}{z \pm ia}\, dz.$$

We want to show that the last integral approaches 0 as $R \to \infty$. Switching to polar coordinates makes life easier.

$$
\begin{aligned}
\int_{|z|=R, y>0} \frac{e^{iz}}{z \pm ia}\, dz &\le \int_0^{\pi} \left| \frac{e^{iRe^{i\theta}}}{Re^{i\theta} \pm ia} \right| R\, d\theta \\
&= \int_0^{\pi} \left| \frac{e^{iR(\cos\theta + i\sin\theta)}}{Re^{i\theta} \pm ia} \right| R\, d\theta \\
&= \int_0^{\pi} \left| \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{e^{i\theta} \pm ia/R} \right| d\theta
\end{aligned}
$$

Since $|e^{ix}| = 1$ for any real x, the dominating factor in the integrand is $e^{-R\sin\theta}$, making the integral go to 0 as $R \to \infty$.

In $\int_C \frac{e^{iz}}{z+ia}\, dz$, the integrand has no poles in the upper half-plane, so the integral is zero. In $\int_C \frac{e^{iz}}{z-ia}\, dz$, the integrand has a simple pole in $z = ia$ with residue $e^{i(ia)} = e^{-a}$, so for any R large enough, the integral is $2\pi i e^{-a}$ by the residue integration method. Letting $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x+ia}\,dx = 0$$
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x-ia}\,dx = 2\pi i e^{-a}.$$

We add these integrals together to find the second integral in the problem.

$$\begin{aligned}
2\pi i e^{-a} &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x+ia}\,dx + \int_{-\infty}^{\infty} \frac{e^{ix}}{x-ia}\,dx \\
&= \int_{-\infty}^{\infty} \left(\frac{e^{ix}}{x+ia} + \frac{e^{ix}}{x-ia} \right) dx \\
&= \int_{-\infty}^{\infty} \frac{(x+ia+x-ia)e^{ix}}{(x+ia)(x-ia)}\,dx \\
&= \int_{-\infty}^{\infty} \frac{2x(\cos x + i\sin x)}{x^2+a^2}\,dx \\
&= \int_{-\infty}^{\infty} \frac{2xi\sin x}{x^2+a^2}\,dx
\end{aligned}$$

Here we removed the real part in the last step, since we know that it is zero. We get

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \pi e^{-a}.$$

b We use the hint and look for a mapping of the form

$$f(z) = w = \frac{az+b}{cz+d}.$$

Use the given information:

$$\begin{aligned}
-1 = f(0) &= \frac{b}{d} \\
\implies d &= -b \\
i = f(i) &= \frac{ia+b}{ic-b} \\
\implies -c - ib &= ia + b \\
-i = f(-i) &= \frac{-ia+b}{-ic-b} \\
\implies -c + ib &= -ia + b
\end{aligned}$$

By combining the equations, we get $-2c = 2b$ and $-2ib = 2ia$. That is, $a = -b = c$, which gives

$$\begin{aligned}
f(z) &= \frac{az-a}{az+a} \\
&= \frac{z-1}{z+1}.
\end{aligned}$$

The imaginary axis is sent to the unit circle, with the point at infinity being sent to 1. (Observe that $|z-1| = |z+1|$ for imaginary z.) Thus the right half-plane is either sent onto the unit disc, or the outside of the unit circle. $f(1) = 0$ is inside the disc, so the first alternative is correct one.