$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}}\, e^{-\frac{\omega^2}{4a}} \qquad (a>0)$$

We calculate this, for simplicity, when $a = 1/2$.

$$F(\omega) \overset{\text{DEF.}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-i\omega x}\, dx$$

DIFFERENTATE $\frac{d}{d\omega}$

$\frac{d}{d\omega}$

$$F'(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left(-x e^{-\frac{x^2}{2}}\right)}_{= \frac{d}{dx} e^{-x^2/2}} i e^{-i\omega x}\, dx$$

[INTEGRATION BY PARTS]

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \Big[e^{-x^2/2}\, i\, e^{-i\omega x} \Big]_{-\infty}^{\infty}}_{=0} + \underbrace{\frac{i^2\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-i\omega x}\, dx}_{\text{! Here } F(\omega) \text{ appears again!}}$$

$|ie^{-i\omega x}| \equiv 1$

$$= -\omega F(\omega)$$

!

Thus

$$\frac{dF}{d\omega} + \omega F = 0 \iff F(\omega) = Ce^{-\frac{\omega^2}{2}}$$

In order to determine the integration constant, we set $\omega = 0$

$$C \cdot 1 = F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt}_{\text{Important integral.}} = 1$$

It follows that

$$\boxed{\widehat{e^{-\frac{x^2}{2}}} = e^{-\frac{\omega^2}{2}}}$$