

Miscellaneous

- Heaviside function $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- Dirac Delta function $\delta(t-a)$ is zero everywhere except a and satisfies $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, moreover $\int_{-\infty}^{\infty} g(t)\delta(t-a) = g(a)$ for any continuous function g .
- Convolution For functions defined on the real line:
 $f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad -\infty < x < \infty;$
 for functions defined only on the positive half-axis:
 $f * g(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0.$

Laplace transform

$\mathcal{L}\{f\}(s) = F(s) \int_0^{\infty} f(t)e^{-st}dt$	$f(t)$	$F(s)$
$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$	1	$\frac{1}{s}$
$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$	$t^n, n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$	e^{at}	$\frac{1}{s-a}$
$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$	$t^n e^{at}, n = 1, 2, \dots$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$	$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s), c > 0$	$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}\{tf(t)\}(s) = -F'(s)$	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^{\infty} F(\sigma)d\sigma$	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
	$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
	$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- Periodic functions with period $2L$, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x dx$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx$$

- Parseval's identitics $\frac{1}{2L} \int_{-L}^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$, $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$
 - $\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$
 - $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$
 - $\hat{f}'(w) = iw\hat{f}(w)$
 - $\widehat{f''}(w) = -w^2 \hat{f}(w)$
 - $\widehat{f(x-a)}(w) = e^{-iaw} \hat{f}(w)$
 - $\widehat{f(w-b)} = e^{ibw} \widehat{f(x)}(w)$
 - $\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g}$
- | $f(x)$ | $\hat{f}(w)$ |
|---|--|
| $\delta(x-a)$ | $\frac{1}{\sqrt{2\pi}} e^{-iaw}$ |
| $\begin{cases} 1, & -b \leq x \leq b \\ 0, & x > b \end{cases}$ | $\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$ |
| $e^{-ax} u(x)$ | $\frac{1}{\sqrt{2\pi(a+iw)}}$ |
| $\frac{1}{x^2+a^2}$ | $\sqrt{\frac{\pi}{2}} \frac{e^{-a w }}{a}$ |
| e^{-ax^2} | $\frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$ |

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$,
 $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cosh z = \frac{e^z + e^{-z}}{2}$, $\sinh z = \frac{e^z - e^{-z}}{2}$

- Taylor and Laurant series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad b_n = \frac{1}{2\pi i} \oint_C f(z) (z - z_0)^{n-1} dz$$

- Let $f(z)$ have pole of order n at z_0 . Then

$$\text{Res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)].$$

In particular

$$\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} [(z - z_0)f(z)] \quad \text{if } n = 1;$$

$$\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} \frac{d}{dz} [(z - z_0)^2 f(z)] \quad \text{if } n = 2;$$

Some useful integrals

- $\int x \sin ax dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
- $\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
- $\int x^2 \sin ax dx = \frac{2}{a^2} x \sin ax + \frac{2 - a^2 x^2}{a^3} \cos ax + C$
- $\int x^2 \cos ax dx = \frac{2}{a^2} x \cos ax - \frac{2 - a^2 x^2}{a^3} \sin ax + C$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$
- $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \quad a > 0.$