NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4120 Calculus 4K**

Academic contact during examination: Yurii Lyubarskii[a], Peter Lindqvist[b]

Phone: [a]91647362 , [b]73593520

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Examination time (from–to): 09:00–13:00

Permitted examination support material: C: A specific basic calculator is allowed.

Other information:
The eight problems 1a, 1b, 2, 3a, 3b, 4, 5, 6 have equal weights. At the end of the text you find auxiliary formulas.

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Informasjon om trykking av eksamensoppgave
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Date Signature

Problem 1

a) Find the inverse Laplace transform

$$f(t) = \mathfrak{L}^{-1}\left\{\frac{s^2+1}{s^2(s^2-4s+9)}\right\}.$$

b) Solve the initial value problem

$$\begin{cases} y''(t) - 4y'(t) + 9y(t) = t, \quad t > 0; \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Problem 2

The equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0,$$

describes an infinite vibrating string. The initial values are

$$u(x,0) = \begin{cases} x+1, & -1 < x < 0; \\ 1-2x, & 0 < x < \frac{1}{2}; \\ 0 & \text{otherwise}, \end{cases}$$

and the initial speed is $u_t(x,0) = 0$. Sketch the shape of the string at time $t = \frac{3}{2}$.

Problem 3

a) Expand

$$g(x) = x^2 - \pi x, \qquad 0 < x < \pi,$$

in a Fourier sine-series.

b) Solve the equation

$$u_{xx} + u_{yy} - 20u = 0$$

in the rectangle

$$0 < x < \pi,\ 0 < y < 1.$$

The boundary values are

$$\begin{cases} u(0,y) = 0,\ u(\pi,y) = 0, \qquad 0 < y < 1; \\ u(x,0) = 0,\ u(x,1) = x^2 - \pi x, \qquad 0 < x < \pi. \end{cases}$$

Hint: Separate the variables.

Problem 4 Find the Laurent expansion around the point $z = i$ of the function

$$f(z) = \frac{1}{z^2 + 1}.$$

In which domain does the expansion converge?

Problem 5 Construct a conformal mapping $w = w(z)$ of the domain

$$-\frac{\pi}{4} < \arg(z) < +\frac{\pi}{4}$$

onto the half-plane $\mathrm{Im}(w) > 0$ such that $w(1) = 2i$.

Problem 6 Find the Fourier transform $\hat{f}(\omega)$ of the function

$$f(x) = \frac{1}{(x^2 + 1)(x^2 + 4)}.$$

Hint: The calculation of $\hat{f}(\omega)$ for $\omega > 0$ and $\omega < 0$ should be done separately.

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero everywhere except a and satisfies $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, moreover $\int_{-\infty}^{\infty} g(t)\delta(t-a) = g(a)$ for any continuous function g.
- **Convolution** For functions defined on the real line:
  $f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad -\infty < x < \infty;$
  for functions defined only on the positive half-axis:
  $f * g(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0.$

Laplace transform

- $\mathcal{L}\{f\}(s) = F(s) \int_0^{\infty} f(t)e^{-st}dt$
- $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
- $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\left\{\int_0^t f(\tau)d\tau)\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$
- $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
- $\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$, $c > 0$
- $\mathcal{L}\{tf(t)\}(s) = -F'(s)$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^{\infty} F(\sigma)d\sigma$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- Periodic functions with period $2L$, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

- Parseval's identities $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$, $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$
- $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$
- $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$
- $\widehat{f'}(w) = iw\hat{f}(w)$
- $\widehat{f''}(w) = -w^2\hat{f}(w)$
- $\widehat{f(x-a)}(w) = e^{-iaw}\hat{f}(w)$
- $\hat{f}(w-b) = \widehat{e^{ibx}f(x)}(w)$
- $\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$

$f(x)$	$\hat{f}(w)$
$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\begin{cases}1, & -b \le x \le b\\ 0, & \lvert x\rvert > b\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i\sin y)$,
  $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \; a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \ b_n = \frac{1}{2\pi i} \oint_C f(z)(z - z_0)^{n-1} dz$$

- Let $f(z)$ have a pole of order n at z_0. Then

$$\operatorname*{Res}_{z=z_0}\{f\} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)].$$

In particular

$$\operatorname*{Res}_{z=z_0}\{f\} = \lim_{z \to z_0} [(z - z_0) f(z)] \qquad \text{if } n = 1;$$

$$\operatorname*{Res}_{z=z_0}\{f\} = \lim_{z \to z_0} \frac{d}{dz} [(z - z_0)^2 f(z)] \qquad \text{if } n = 2.$$

Some useful integrals

- $\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
- $\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
- $\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2 - a^2x^2}{a^3} \cos ax + C$
- $\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2 - a^2x^2}{a^3} \sin ax + C$
- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + C$
- $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + C$
- $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \ a > 0$