TMA 4120 13. VIII. 2019 P. Lindqvist

Matem. 4K

① $\begin{cases} y'' - 4y' + 13y = 4\,\delta(t-3) \\ y(0) = 0 = y'(0) \end{cases}$

$$(s^2 - 4s + 13)Y(s) = 4e^{-3s}$$

$$Y(s) = 4e^{-3s}\frac{1}{(s-2)^2+3^2}$$ (s-shift)

$$= \frac{4}{3}e^{-3s}\mathcal{L}\{e^{2t}\sin(3t)\}$$ (t-shift)

$$= \frac{4}{3}\mathcal{L}\{e^{2(t-3)}\sin(3(t-3))\,u(t-3)\}$$

$$y(t) = \frac{4}{3}u(t-3)\,e^{2(t-3)}\sin(3(t-3))$$

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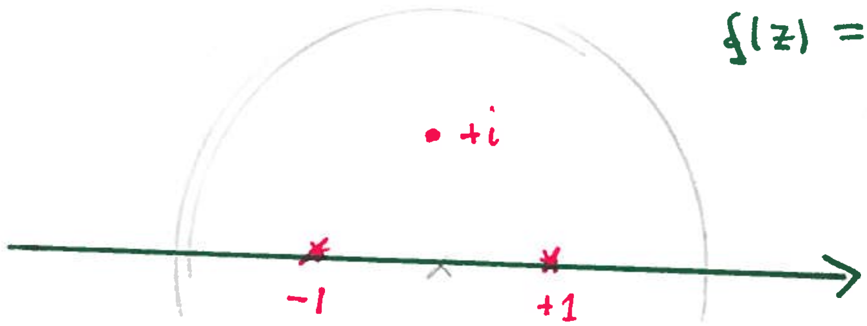
② $$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \cdot 3 \longrightarrow 3 \quad \text{as } n \to \infty$$

Hence $R = \frac{1}{3}$ by a formula for the radius of convergence.

(3) Simple poles at $x = \pm 1$ (on the line of integration) and at $z = \pm i$.

$$f(z) = \frac{1}{(z^2-1)(z^2+1)}$$



$$PV\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)(x^2+1)} = 2\pi i \operatorname*{Res}_{z=i}\{f(z)\}$$

($z = -i$ not included!)

$$+ \frac{2\pi i}{2}\left\{\operatorname*{Res}_{z=-1}\{f(z)\} + \operatorname*{Res}_{z=1}\{f(z)\}\right\}$$

Only 50% of the residues on the real axis.

$\underline{z = i}$ $$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} \frac{\cancel{z-i}}{(z^2-1)\cancel{(z-i)}(z+i)} = -\frac{1}{4i}$$

$\underline{z = \pm 1}$ $$\operatorname*{Res}_{z=\pm 1} f(z) = \cdots = \pm\frac{1}{4}.$$

$$PV\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)(x^2+1)} = -\frac{2\pi i}{4i} + 0 = -\frac{\pi}{2}$$

<u>Answer</u> : $-\pi/2$.

④ $\begin{cases} u_t + u = u_{xx}, & u = u(x,t); \quad \underline{0 < x < \pi,\ t > 0} \\ u(0,t) = 0 = u(\pi,t), & t > 0 \end{cases}$

Separation of variables

$$u(x,t) = X(x)\,T(t), \qquad X(0) = X(\pi) = 0$$

$$X\dot{T} + XT = X''T$$

$$\frac{\dot{T}}{T} + 1 = \frac{X''}{X} = \lambda \text{ (constant of separation)}$$

$\begin{cases} X'' - \lambda X = 0 \\ \dot{T} = (\lambda - 1)T \end{cases}$ Only $\lambda = -n^2$ yields solutions $X(x)$ that satisfy $X(0) = X(\pi) = 0$.

$$X(x)T(t) = a_n e^{-(n^2+1)t} \sin(nx)$$

<u>Superposition</u>

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n^2+1)t} \sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) \overset{?}{=} \sum_{n=1}^{\infty} (-1)^n \frac{\sin(2nx)}{n^2}$$

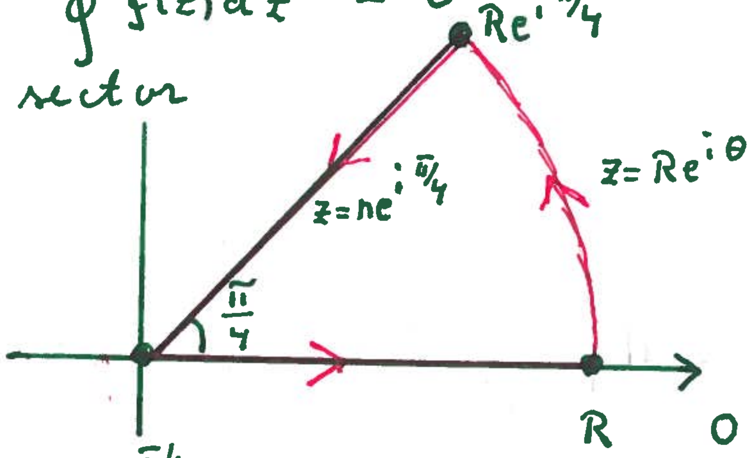
$(a_{2n+1} = 0)$

<u>Answer</u>: $u(x,t) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-(4n^2+1)t} \sin(2nx)$

(5) Let $f(z) = e^{iz^2}$ and consider

$$\oint f(z)\,dz = 0$$

along the sector



$$0 = \int_0^R e^{ix^2}\,dx + \int_0^{\pi/4} e^{i(Re^{i\theta})^2} Rie^{i\theta}\,d\theta + \underbrace{\int_R^0 e^{ir^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}}\,dr}_{-\frac{1+i}{\sqrt{2}}\int_0^R e^{-r^2}\,dr}$$

As $R \to \infty$, the integral along the circle vanishes:

$$\left| \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} Rie^{i\theta}\,d\theta \right| \leq R \int_0^{\pi/4} e^{-R^2 \sin(2\theta)}\,d\theta \xrightarrow[R\to\infty]{} 0.$$

Hence, using $e^{ix^2} = \cos(x^2) + i\sin(x^2)$,

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx = \frac{1+i}{\sqrt{2}} \int_0^\infty e^{-r^2}\,dr = \frac{1+i}{2\sqrt{2}}\sqrt{\pi}$$

<u>Answer:</u>

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

(6a) $$f(z) = \frac{1 + e^{i\pi z}}{(z-1)^2(z+1)^2}$$

Simple poles at $z = \pm 1$.

$$e^{i\pi z} = e^{i\pi(z \pm 1)} e^{\mp i\pi}$$
$$= -1\left[1 + i\pi(z \pm 1) - \frac{\pi^2}{2!}(z \pm 1)^2 + \cdots\right]$$
$$1 + e^{i\pi z} = -(z \pm 1)\left[i\pi - \frac{\pi^2}{2!}(z \pm 1) + \cdots\right]$$

The zeros ± 1 are simple zeros.

(6b) $\underline{z = 1}$ $$f(z) = \frac{-i\pi + \frac{\pi^2}{2!}(z-1)}{(z-1)(z+1)^2}$$

$$\operatorname*{Res}_{z=1}\{f(z)\} = \lim_{z \to 1} (z-1) f(z) = \frac{-i\pi}{(1+1)^2}$$
$$= -\frac{i\pi}{4}$$

$$\operatorname*{Res}_{z=-1}\{f(z)\} = \cdots = -\frac{i\pi}{4}$$