Problems on conformal mappings. (Make pictures!)
(some of them may be complicated 😊)

1. a. Find the image of the circle \( \{ z : |z - \frac{1}{2}| = \frac{1}{2} \} \) under the mapping

\[
z \rightarrow w(z) = \frac{1}{z - 1} \quad (\star)
\]

b. Find the image of the disk \( \{ z : |z - \frac{1}{2}| < \frac{1}{2} \} \) under the mapping (\star)

c. Find the image of the exterior of the disk: \( \{ z : |z - \frac{1}{2}| > \frac{1}{2} \} \) under the mapping (\star)

d. Find the image of the domain between two circles \( \{ z : \frac{1}{2} < |z - \frac{1}{2}| \text{ and } 1 > |z - 1| \} \) under the mapping (\star)

2. Find a conformal mapping of the \( \mathbb{C}_+ \) (upper half-plane) onto

a) \( \{ w, w \in (-\infty, 0] \} \)

b) \( \{ w, \text{ Re} w > 0; w \notin [0, 1] \} \)

3. Find a conformal mapping of the unit disk onto itself such that \( f(0) = \frac{1}{2} \).

a) Find the images of the following curves under the mapping \( z \rightarrow w = z^2 \):
- \( \{ z : \arg z = \alpha \} \) for some fixed \( \alpha \in (\pi, \frac{3\pi}{2}) \)
- \( \{ z : \text{ Re } z = a \} \) for some fixed \( a \in \mathbb{R} \)
8) Find the images under the mapping $z \rightarrow w = z^2$ of the following domains:

- $\Delta 2$: $\text{Im } z > 0$
- $\Delta 2$: $|z| < 2$
- $\Delta 2$: $\text{Re } z > 1$
- $\Delta 2$: $|z| > \frac{1}{2}$, $\text{Re } z > 0$

5) Let the value of the function $w = z^{1/2}$ is chosen by the relation $z^{1/2} \big|_{z=-1} = -i$.

Find the images of the domains

- $\Delta 2$: $z \notin [0,100]$
- $\Delta 2$: $\text{Im } z^2 > 2 \text{Re } z + 1$

6. Find a conformal mapping of the upper half plane $C_+$ onto the strip $\Delta 2$: $|\text{Re } z| < \frac{1}{2}$

7. Find a conformal mapping of $C_+$ onto the half-strip $\Delta 2$: $|\text{Re } z| < 1$, $\text{Im } z > 0$

8) Find a conformal mapping of the disk with a cut: $\Delta 2$: $|z| < 1$, $z \in [i, i]$ onto the disk $\Delta 2$: $|w| < 1$

9) Find a mapping of the unit disk $D = \{ z : |z| < 1 \}$ onto the plane with infinite cut $\{ z \in \mathbb{C} : |z| = 2 \}$, $z \in [0, \infty)$.
10) Find a mapping of the unit disk onto the complex plane with finite cut \( \overline{\mathbb{C}} \setminus \{1, 2\} = \{w \in \mathbb{C}, w \in [1, 2]\} \). 

Comment: When we write \( \overline{\mathbb{C}} \) we add infinity to the complex plane, so \( \overline{\mathbb{C}} \setminus \{1, 2\} \) is a domain without holes.