THE PRINCIPLE OF ARGUMENT

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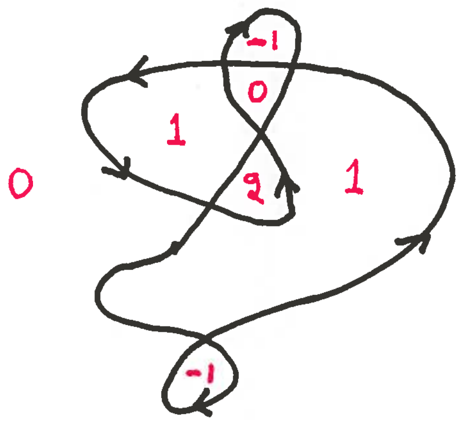
The Principle of Argument counts the number of zeros (with multiplicity) that an analytic function has inside a loop (a closed simple curve).

The **winding number**

$$N = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-z_0} = \text{the number of times the curve } \gamma \text{ winds around the point } z_0.$$



The winding number is constant in each component. It can be negative. Of course, it is an INTEGER.

Consider (for simplicity) a polynomial

$$P(z) = a(z-z_1)(z-z_2)\cdots(z-z_n)$$

with roots $z_1, z_2, \ldots, z_n$. Then

$$\frac{P'(z)}{P(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n}$$

by "logarithmic differentiation". If the loop γ avoids the zeros, we have

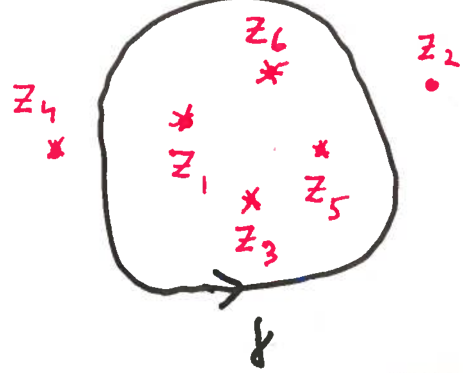
$$\frac{1}{2\pi i}\oint_{\gamma} \frac{P'(z)}{P(z)}\,dz = \frac{1}{2\pi i}\sum_{k=1}^{n}\oint_{\gamma}\frac{dz}{z-z_k}$$

$=$ the number of zeros z_k <u>inside</u> the loop

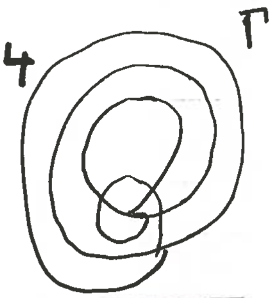
Indeed,

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{dz}{z-z_k} = \begin{cases} 1, & \text{if } z_k \text{ is inside} \\ 0, & \text{if } z_k \text{ is outside}\end{cases}$$

<u>Ex.:</u>



$$\frac{1}{2\pi i}\oint_{\gamma}\frac{P'(z)\,dz}{P(z)} = 4$$



γ has the parametrization $z = z(t)$, say.

Γ is the <u>image curve</u> $w = w(t) = P(z(t))$ in the w-plane.

$$N_\Gamma = \frac{1}{2\pi i}\oint_{\Gamma}\frac{dw}{w-0}$$ $=$ the number of times Γ winds around the <u>origin</u> in the w-plane.

Now

$$dw = P'(z(t))\,dz(t)$$
$$= P'(z(t))\,z'(t)\,dt$$

and so

$$N_\Gamma = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i}\int \frac{P'(z(t))\, z'(t)\, dt}{P(z(t))}$$

$$= \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{P'(z)\, dz}{P(z)}}_{\text{The number of zeros inside the loop } \gamma}$$

PRINCIPLE OF ARGUMENT. The number of zeros z_k encircled by the loop $z = z(t)$ is equal to the number of times the image curve $w(t) = P(z(t))$ winds around the origin (in the w-plane).

<u>Remark:</u> The Principle of Argument is valid for analytic functions $f(z)$, not necessarily polynomials.

Note that no integral has to be calculated when the Principle is used!

The roots are unknown!

$P(z) = z^3 + z^4 + z^5$

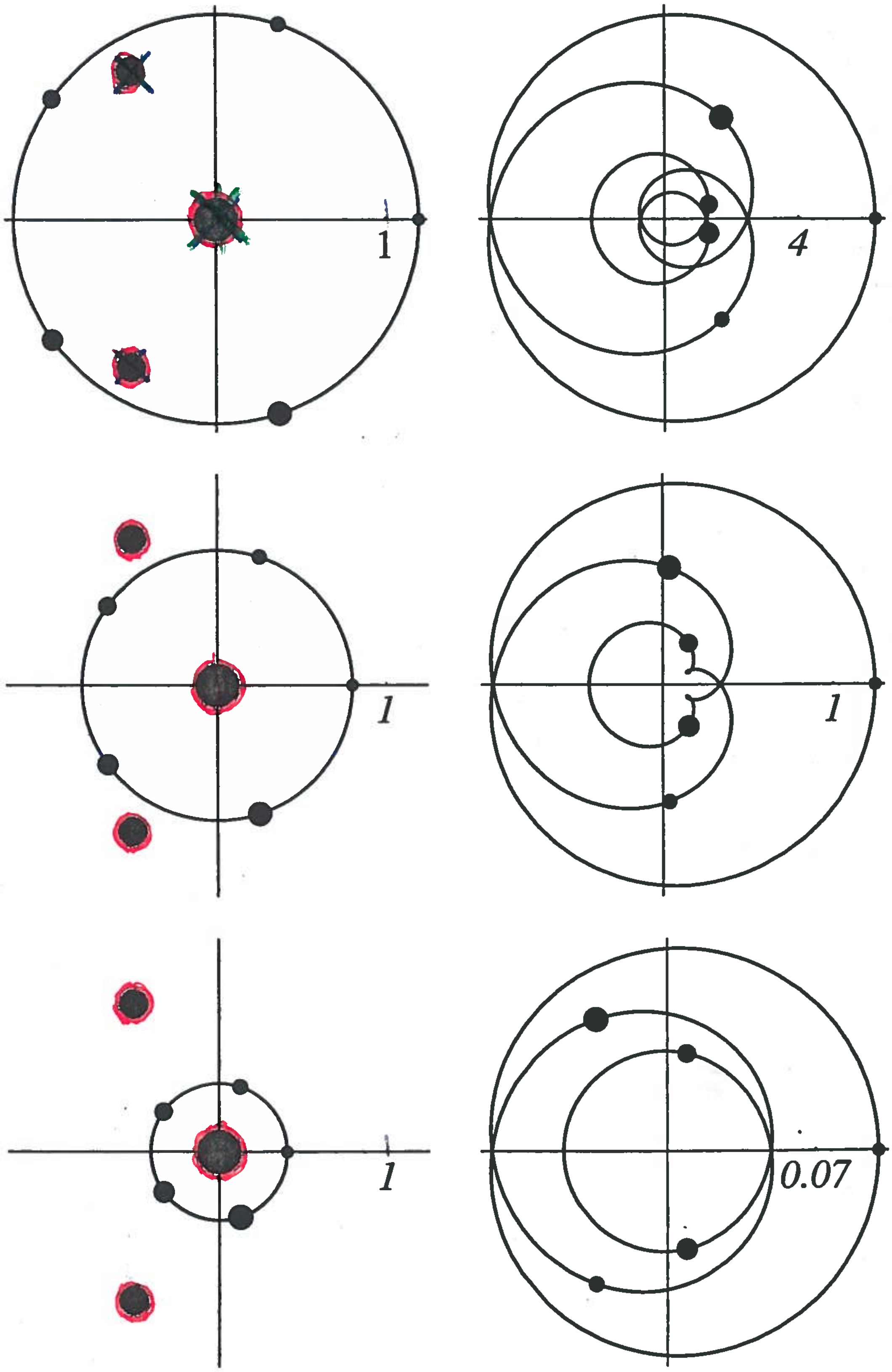


Figure 3.5-2 On the left, three circles $|z| = r$ ($r = 1.2, 0.8, 0.4$). To the right of each is its image γ_r under the mapping $z \mapsto z^3 + z^4 + z^5$. The grey dots indicate the zeros of the function. The zero at the origin is of order 3.