Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4120 Calculus 4K**

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Permitted examination support material: Kode C: approved simple calculator

Other information:
This examination paper contains six problems with nine questions all together. First part of the exam is on Laplace and Fourier transforms and their applications, the second part is on complex analysis. All answers should be justified.

All nine questions are counted equally. Good luck!

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Problem 1

a) Find the inverse Laplace transform of

$$\frac{1}{(s+2)(s+1)}.$$

Solution We have

$$\frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Using the table, we compute the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{(s+2)(s+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right) = e^{-t} - e^{-2t}.$$

b) Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-1), \quad t \geq 0,\ y(0) = 0,\ y'(0) = 0,$$

where δ is the Dirac delta function.

Solution Taking the Laplace transform of both sides, we get,

$$(s^2 + 3s + 2)Y(s) = e^{-s}.$$

Using the result of part a) and the second shift theorem, we obtain

$$y(t) = (e^{-(t-1)} - e^{-2(t-1)})u(t-1) = \begin{cases} 0,\ t < 1 \\ e^{1-t} - e^{2-2t},\ t \geq 1 \end{cases}$$

Problem 2 Consider the boundary value problem for the wave equation:

$$u_{tt} = c^2 u_{xx},\ t > 0,\ 0 < x < \pi, \quad u(t,0) = u(t,\pi) = 0. \quad (*)$$

a) Find all solutions of $(*)$ on the form $u(t,x) = F(t)G(x)$.

Solution We divide the variables and get

$$\frac{F''(t)}{F(t)} = c^2 \frac{G''(x)}{G(x)} = k.$$

Taking into account the boundary condition, we have $G(0) = G(\pi) = 0$. We have a solution of the equation $G'' = kc^{-2}G$ which is equal to zero at two points. Then $kc^{-2} < 0$ and if we denote $kc^{-2} = -p^2$ then

$$G_p(x) = a_p \cos px + b_p \sin px.$$

The condition $G(0) = G(\pi) = 0$ implies $a_p = 0$ and $p = 1, 2, ...$ is integer. Thus $k = -(cp)^2$ and

$$G_p(x) = b_p \sin px, \quad p = 1, 2, ...$$

Then $F_p'' = -(cp)^2 F_p$ and $F_p(t) = (A_p \cos cpt + B_p \sin cpt)$. Finally, renaming the constants, we get all solutions with separated variables of the form:

$$u_p(t, x) = \sin px(A_p \cos cpt + B_p \sin cpt). \qquad (**)$$

b) Find the solution of $(*)$ that also satisfies the following initial condition

$$u(0, x) = \pi x - x^2, \quad u_t(0, x) = 0, \quad 0 < x < \pi.$$

Solution We are looking for a solution $u(t, x)$ of the form

$$u(t, x) = \sum_p u_p(t, x),$$

where $u_p(x, t)$ is a solution of the form (**). The initial conditions give

$$u(0, x) = \sum_p A_p \sin px, \quad u_t(0, x) = \sum_p pcB_p \sin px.$$

Then $B_p = 0$ and A_p are sine-Fourier coefficients of the function $f(x) = \pi x - x^2$. We extend $f(x)$ in an odd way to $(-\pi, \pi)$ and find

$$A_p = \frac{2}{\pi} \int_0^\pi (\pi x - x^2) \sin px dx = -\frac{2}{p\pi} \int_0^\pi (\pi x - x^2)(\cos px)' dx =$$
$$\frac{2}{p\pi} \int_0^\pi (\pi - 2x) \cos px dx =$$
$$\frac{4}{p^2\pi} \int_0^\pi \sin px dx = \frac{4}{p^3\pi}(1 - (-1)^p).$$

Finally,

$$u(t, x) = \sum_{p=1}^{\infty} \frac{4(1 - (-1)^p)}{p^3\pi} \cos cpt \sin px.$$

Problem 3 Calculate the Fourier transform of the function $f(t) = e^{-3t} \sin bt\, u(t)$, where $u(t)$ is the Heaviside step function.

Solution We note first that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. Then we have

$$\hat{f}(w) = \frac{1}{2i\sqrt{2\pi}} \int_0^\infty e^{-3t}(e^{ibt} - e^{-ibt})e^{-itw} dt$$

Using the Fourier transform of the function $u(t)e^{at}$ from the table, we get

$$\hat{f}(w) = \frac{1}{2i\sqrt{2\pi}} \left(\frac{1}{3 + i(w-b)} - \frac{1}{3 + i(w+b)} \right) = \frac{b}{\sqrt{2\pi}} \frac{1}{(3+iw)^2 + b^2}.$$

Problem 4 Let $f(z) = z^{-3}e^{z^2}$.

a) Find the Laurent series of f centered at the origin. Where does the series converge? What type of singularity has f at the origin?

Solution We start with the series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Then

$$z^{-3}e^{z^2} = z^{-3} \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = z^{-3} + z^{-1} + \sum_{n=2}^{\infty} \frac{z^{2n-3}}{n!}$$

This is the Laurent series of f centered at the origin, it converges when $z \neq 0$, f has a pole (or order 3) at the origin.

b) Compute

$$\oint_C f(z)dz,$$

along the unit circle, $|z| = 1$.

Solution Clearly, f has only one singularity inside the unit circle, the origin. By the residue theorem

$$\oint_C f(z)dz = 2\pi i Res_{z=0} f(z).$$

From the Laurent series in a) we see that $Res_{z=0} f(z) = 1$. Thus

$$\oint_C f(z)dz = 2\pi i.$$

Problem 5 The analytic function $f(z)$ satisfies

$$\frac{\partial f}{\partial y} = 2z, \ f(0) = 1.$$

Find the function $f(z)$.

Solution We mean as usual that $z = x + iy$. Let $f(z) = u(x, y) + iv(x, y)$. Then we have

$$u_y + iv_y = 2x + i2y.$$

It means that $u_y = 2x$ and $v_y = 2y$. We combine that with the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$. We get

$$u_x = 2y, \ u_y = 2x, \quad v_x = -2x, \ v_y = 2y.$$

We integrate and obtain

$$u(x, y) = 2xy + c_1, \quad v(x, y) = y^2 - x^2 + c_2.$$

Then $f(z) = 2xy + i(y^2 - x^2) + c = -iz^2 + c$. The condition $f(0) = 1$ implies $c = 1$. Thus

$$f(z) = -iz^2 + 1.$$

Problem 6 Evaluate the integral

$$\int_0^\infty \frac{x^2 dx}{x^4 + 16}.$$

Solution First we note that the function under the integral is even and thus

$$\int_0^\infty \frac{x^2 dx}{x^4 + 16} = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 dx}{x^4 + 16}$$

In the last integral the degree of the polynomial $x^4 + 16$ is large then the degree of x^2 by two and by the residue formula we get

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 + 16} = 2\pi i \sum_{z_j} Res_{z=z_j} \frac{z^2}{z^4 + 16}.$$

Where the sum is taken over all singular points in the upper half-plane. We find the roots of the equation $z^4 + 16 = 0$. There are two roots in the upper half-plane:

$z_1 = \sqrt{2} + i\sqrt{2}$ and $z_2 = -\sqrt{2} + i\sqrt{2}$. The formula for the residue at simple pole gives

$$Res_{z=z_j} f(z) = \frac{z_j^2}{4z_j^3} = \frac{1}{4z_j}.$$

Then the sum of the residues is

$$Res_{z=z_1} f(z) + Res_{z=z_2} f(z) = \frac{z_1 + z_2}{4z_1 z_2} = \frac{-i2\sqrt{2}}{16} = \frac{-i\sqrt{2}}{8}.$$

Finally,

$$\int_0^\infty \frac{x^2}{x^4+16} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{x^4+16} dx = \pi i \sum_j Res_{z=z_j} \frac{z^2}{z^4+16} = \frac{\pi\sqrt{2}}{8}.$$

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero everywhere except a and satisfies $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, moreover $\int_{-\infty}^{\infty} g(t)\delta(t-a) = g(a)$ for any continuous function g.
- **Convolution** For functions defined on the real line:
  $f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad -\infty < x < \infty;$
  for functions defined only on the positive half-axis:
  $f * g(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0.$

Laplace transform

- $\mathcal{L}\{f\}(s) = F(s) \int_0^{\infty} f(t)e^{-st}dt$
- $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
- $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$
- $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
- $\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$, $c > 0$
- $\mathcal{L}\{tf(t)\}(s) = -F'(s)$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^{\infty} F(\sigma)d\sigma$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- Periodic functions with period $2L$, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

- Parseval's identities $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$, $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$
- $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$
- $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$
- $\hat{f'}(w) = iw\hat{f}(w)$
- $\widehat{f''}(w) = -w^2\hat{f}(w)$
- $\widehat{f(x-a)}(w) = e^{-iaw}\hat{f}(w)$
- $\hat{f}(w-b) = \widehat{e^{ibx}f(x)}(w)$
- $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$

$f(x)$	$\hat{f}(w)$
$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\begin{cases}1, & -b \le x \le b \\ 0, & \lvert x\rvert > b\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i\sin y)$,
  $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurant series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \ a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}, \ b_n = \frac{1}{2\pi i}\oint_C f(z)(z-z_0)^{n-1}dz$$

Some useful integrals

- $\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
- $\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
- $\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3} \cos ax + C$
- $\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$
- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$
- $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$
- $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \ a > 0$