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Examination paper for **TMA4120 Calculus 4K**

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Examination date: one day in November-December

Examination time (from–to): 4 hours

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood! All the formulas you may need are in the attachment, see the last pages of the exam paper. However a simple calculator is still allowed on the final exam.

Other information:
This examination paper contains six problems with ten parts all together. Approximately a half of them is on Laplace transform, Fourier series and transform and their applications to differential equations and half on complex analysis. The problem parts are counted equally. Good luck!

Language: English

Number of pages: 1

Number of pages enclosed: 3

Checked by:

Date Signature

Problem 1 Solve the initial value problem

$$y'' + 4y' + 4y = u(t-1), \quad y(0) = 0, \quad y'(0) = 1.$$

Problem 2 We consider the following boundary value problem

$$u_{tt} + 2u_t + u = c^2 u_{xx}, \ 0 < x < L, \ t > 0, \quad u(0,t) = u(L,t) = 0. \qquad (*)$$

a) Find all solutions of $(*)$ of the form $u(x,t) = F(x)G(t)$.

b) Compute the coefficients of the sine Fourier series $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ of the function $f(x)$ defined by $f(x) = x(L-x)$ for $0 < x < L$.

c) Find a solution of $(*)$ that satisfies the initial conditions

$$u(x,0) = f(x), \ u_t(x,0) = 0,$$

where f is a function defined in b).

Problem 3 Show that if $\hat{f}(w)$ is the Fourier transform of a function $f(x)$ then the Fourier transform of the function $f(x)\sin bx$ is equal to $\frac{i}{2}(\hat{f}(w+b) - \hat{f}(w-b))$. Compute the Fourier transform of the function $g(x) = \begin{cases} \sin 2x, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Problem 4 Let $z_0 = i$. Determine the real and imaginary parts of

$$1 + z_0; \ \frac{1+z_0}{1-z_0}; \ \mathrm{Ln} z_0; \ \mathrm{Ln}(1+z_0).$$

Problem 5 Let $f(z) = \frac{\cos z}{z^2+1}$.

a) Find all zeros and singular points of $f(z)$, classify the singularities.

b) Compute $\oint_C f(z)dz$, where C is the circle of radius 3 centered at $z_0 = 1$.

c) Let $g(w) = f(1/w)$ show that $g(w)$ has essential singularity at zero. (Hint: consider the values of g on a set $0 < |w| < r$.)

Problem 6 Evaluate the integral $\int_0^{\infty} \frac{\cos 3x}{x^4+3x^2+2} dx$.

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero everywhere except a and satisfies $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, moreover $\int_{-\infty}^{\infty} g(t)\delta(t-a) = g(a)$ for any continuous function g.
- **Convolution** For functions defined on the real line:
  $f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \ -\infty < x < \infty$;
  for functions defined only on the positive half-axis:
  $f * g(x) = \int_0^x f(y)g(x-y)dy$.

Laplace transform

- $\mathcal{L}\{f\}(s) = F(s) \int_0^\infty f(t)e^{-st}dt$
- $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
- $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$
- $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
- $\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$, $c > 0$
- $\mathcal{L}\{tf(t)\}(s) = -F'(s)$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)d\sigma$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- Periodic functions with period $2L$, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

- Parseval's identities
$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$, $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$
- $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$
- $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$
- $\widehat{f'}(w) = iw\hat{f}(w)$
- $\widehat{f''}(w) = -w^2\hat{f}(w)$
- $\widehat{f(x-a)}(w) = e^{-iaw}\hat{f}(w)$
- $\hat{f}(w-b) = \widehat{e^{ibx}f(x)}(w)$
- $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$

$f(x)$	$\hat{f}(w)$
$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\begin{cases}1, & \lvert x\rvert \le a\\ 0, & \lvert x\rvert > a\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin aw}{w}$
$\begin{cases}e^{-ax}, & x \ge 0\\ 0, & x < 0\end{cases}$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i\sin y)$,
$\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,\ a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n},\ b_n = \frac{1}{2\pi i}\oint_C f(z)(z-z_0)^{n-1}dz$$

Some useful integrals

- $\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
- $\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
- $\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2x}{a^3} \cos ax + C$
- $\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$
- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$
- $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$
- $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \; a > 0$