Problem 1. Use The Laplace transform to solve the initial value problem

$$y' + y + \int_0^t y(\tau)e^{t-\tau}d\tau = u(t-1), \quad t > 0, \quad y(0) = 1$$

Problem 2. The 2π-periodic function f is defined by $f(x) = e^x, \ -\pi < x < \pi$.
a) Sketch the graph of the periodic extension f and find the complex Fourier series of f.
b) Determine the sums of the series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}, \qquad \sum_{n=2}^{\infty} \frac{1}{1+n^2}$$

Problem 3. Compute the Fourier transform of the function

$$f(x) = \begin{cases} e^{-x}, x \geq 0 \\ e^x, \quad x < 0 \end{cases}.$$

Use the result to compute the integral

$$\int_0^\infty \frac{w \sin w}{1+w^2} dw.$$

Problem 4. a) Find the Fourier Sine Series of the function $f(x) = \pi x - x^2, \ 0 \leq x \leq \pi$.
b) Let $u(x,t)$ be a solution of the boundary value problem

$$\begin{cases} u_t = u_{xx} - 2u_x, & 0 < x < \pi, t > 0, \\ u(0,t) = 0 = u(\pi, t), & t > 0. \end{cases}$$

Show that if $u(x,t) = F(x)G(t)$ then $F(x) = Ce^x \sin nx$ for some integer n.
c) Find a solution $u(x,t)$ of the boundary value problem in b) such that $u(x,0) = e^x f(x), \ 0 < x < \pi$, where $f(x)$ is the function given in a).

Problem 5. The function $f(z) = y^3 + Bx^2y + iv(x,y)$ is analytic. Determine the constant B and the function $v(x,y)$ if $v(0,0) = 0$. (Hint: Use the Laplaces and Cauchy–Riemann equations.)

Problem 6. a) Find all Laurent series with center $z = 0$ of the function $f(z) = (z(8z^3 - 1))^{-1}$ and determine the domain of convergence for each series.
b) Let C be the unit circle $|z| = 1$ with positive orientation (counter clockwise). Determine the values of the integrals

$$\oint_C f(z)dz \qquad \text{and} \qquad \oint_C (\text{Re}z)dz$$

Problem 7. It is given that

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+6x+25} dx = 2\pi i \sum \text{Res} \left\{ \frac{e^{2iz}}{z^2+6z+25} \right\},$$

where the sum is taken over the singular points of the function in the upper half-plane. Compute the integral and determine the value of

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+6x+25} dx.$$

Problem 8. Use the residue calculus to compute the integral

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}.$$