Type	New Variables		Normal Form
Hyperbolic	$v = \Phi$	$w = \Psi$	$u_{vw} = F_1$
Parabolic	$v = x$	$w = \Phi = \Psi$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \Psi)$	$w = \frac{1}{2i}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

Here, $\Phi = \Phi(x, y), \Psi = \Psi(x, y), F_1 = F_1(v, w, u, u_v, u_w)$, etc., and we denote u as function of v, w again by u, for simplicity. We see that the normal form of a hyperbolic PDE is as in d'Alembert's solution. In the parabolic case we get just one family of solutions $\Phi = \Psi$. In the elliptic case, $i = \sqrt{-1}$, and the characteristics are complex and are of minor interest. For derivation, see Ref. [GenRef3] in App. 1.

XAMPLE 1 D'Alembert's Solution Obtained Systematically

The theory of characteristics gives d'Alembert's solution in a systematic fashion. To see this, we write the wave equation $u_{tt} - c^2 u_{xx} = 0$ in the form (14) by setting $y = ct$. By the chain rule, $u_t = u_y y_t = cu_y$ and $u_{tt} = c^2 u_{yy}$. Division by c^2 gives $u_{xx} - u_{yy} = 0$, as stated before. Hence the characteristic equation is $y'^2 - 1 = (y' + 1)(y' - 1) = 0$. The two families of solutions (characteristics) are $\Phi(x, y) = y + x = \text{const}$ and $\Psi(x, y) = y - x = \text{const}$. This gives the new variables $v = \Phi = y + x = ct + x$ and $w = \Psi = y - x = ct - x$ and d'Alembert's solution $u = f_1(x + ct) + f_2(x - ct)$. ■

ROBLEM SET 12.4

Show that c is the speed of each of the two waves given by (4).

Show that, because of the boundary conditions (2), Sec. 12.3, the function f in (13) of this section must be odd and of period $2L$.

If a steel wire 2 m in length weighs 0.9 nt (about 0.20 lb) and is stretched by a tensile force of 300 nt (about 67.4 lb), what is the corresponding speed of transverse waves?

What are the frequencies of the eigenfunctions in Prob. 3?

8 GRAPHING SOLUTIONS

ing (13) sketch or graph a figure (similar to Fig. 291 in . 12.3) of the deflection $u(x, t)$ of a vibrating string ngth $L = 1$, ends fixed, $c = 1$) starting with initial ocity 0 and initial deflection (k small, say, $k = 0.01$).

. $f(x) = k \sin \pi x$

6. $f(x) = k(1 - \cos \pi x)$

. $f(x) = k \sin 2\pi x$

8. $f(x) = kx(1 - x)$

–18 NORMAL FORMS

nd the type, transform to normal form, and solve. Show ur work in detail.

). $u_{xx} + 4u_{yy} = 0$

10. $u_{xx} - 16u_{yy} = 0$

11. $u_{xx} + 2u_{xy} + u_{yy} = 0$

12. $u_{xx} - 2u_{xy} + u_{yy} = 0$

13. $u_{xx} + 5u_{xy} + 4u_{yy} = 0$

14. $xu_{xy} - yu_{yy} = 0$

15. $xu_{xx} - yu_{xy} = 0$

16. $u_{xx} + 2u_{xy} + 10u_{yy} = 0$

17. $u_{xx} - 4u_{xy} + 5u_{yy} = 0$

18. $u_{xx} - 6u_{xy} + 9u_{yy} = 0$

19. **Longitudinal Vibrations of an Elastic Bar or Rod.** These vibrations in the direction of the x-axis are modeled by the wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = E/\rho$ (see Tolstov [C9], p. 275). If the rod is fastened at one end, $x = 0$, and free at the other, $x = L$, we have $u(0, t) = 0$ and $u_x(L, t) = 0$. Show that the motion corresponding to initial displacement $u(x, 0) = f(x)$ and initial velocity zero is

$$u = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n ct,$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin p_n x \, dx, \qquad p_n = \frac{(2n + 1)\pi}{2L}.$$

20. **Tricomi and Airy equations.**[2] Show that the *Tricomi equation* $yu_{xx} + u_{yy} = 0$ is of mixed type. Obtain the **Airy equation** $G'' - yG = 0$ from the Tricomi equation by separation. (For solutions, see p. 446 of Ref. [GenRef1] listed in App. 1.)

[2] Sir GEORGE BIDELL AIRY (1801–1892), English mathematician, known for his work in elasticity. FRANCESCO TRICOMI (1897–1978), Italian mathematician, who worked in integral equations and functional analysis.

12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

After the wave equation (Sec. 12.2) we now derive and discuss the next "big" PDE, the **heat equation**, which governs the temperature u in a body in space. We obtain this model of temperature distribution under the following.

Physical Assumptions

1. The *specific heat* σ and the *density* ρ of the material of the body are constant. No heat is produced or disappears in the body.
2. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient (cf. Sec. 9.7) of the temperature; that is, the velocity $\mathbf{v}$ of the heat flow in the body is of the form

$$\mathbf{v} = -K \operatorname{grad} u \tag{1}$$

where $u(x, y, z, t)$ is the temperature at a point (x, y, z) and time t.

3. The *thermal conductivity* K is constant, as is the case for homogeneous material and nonextreme temperatures.

Under these assumptions we can model heat flow as follows.

Let T be a region in the body bounded by a surface S with outer unit normal vector $\mathbf{n}$ such that the divergence theorem (Sec. 10.7) applies. Then

$$\mathbf{v} \cdot \mathbf{n}$$

is the component of $\mathbf{v}$ in the direction of $\mathbf{n}$. Hence $|\mathbf{v} \cdot \mathbf{n}\, \Delta A|$ is the amount of heat *leaving* T (if $\mathbf{v} \cdot \mathbf{n} > 0$ at some point P) or *entering* T (if $\mathbf{v} \cdot \mathbf{n} < 0$ at P) per unit time at some point P of S through a small portion ΔS of S of area ΔA. Hence the total amount of heat that flows across S from T is given by the surface integral

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dA.$$

Note that, so far, this parallels the derivation on fluid flow in Example 1 of Sec. 10.8.

Using Gauss's theorem (Sec. 10.7), we now convert our surface integral into a volume integral over the region T. Because of (1) this gives [use (3) in Sec. 9.8]

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dA = -K\iint_S (\operatorname{grad} u) \cdot \mathbf{n}\, dA = -K\iiint_T \operatorname{div} (\operatorname{grad} u)\, dx\, dy\, dz \tag{2}$$

$$= -K\iiint_T \nabla^2 u\, dx\, dy\, dz.$$

Here,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

is the **Laplacian** of u.

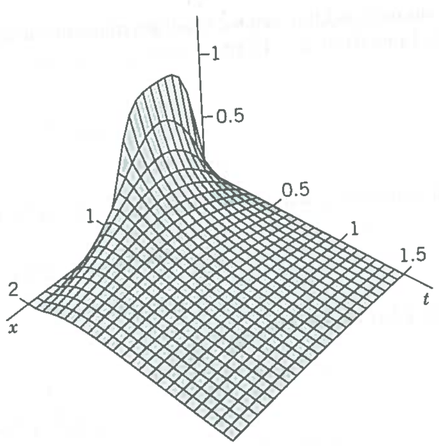

Fig. 300. Solution (20) in Example 4

PROBLEM SET 12.7

1. CAS PROJECT. Heat Flow. (a) Graph the basic Fig. 299.

(b) In (a) apply animation to "see" the heat flow in terms of the decrease of temperature.

(c) Graph $u(x, t)$ with $c = 1$ as a surface over a rectangle of the form $-a < x < a,\quad 0 < y < b$.

2–8 SOLUTION IN INTEGRAL FORM

Using (6), obtain the solution of (1) in integral form satisfying the initial condition $u(x, 0) = f(x)$, where

2. $f(x) = 1$ if $|x| < a$ and 0 otherwise

3. $f(x) = 1/(1 + x^2)$.
Hint. Use (15) in Sec. 11.7.

4. $f(x) = e^{-|x|}$

5. $f(x) = |x|$ if $|x| < 1$ and 0 otherwise

6. $f(x) = x$ if $|x| < 1$ and 0 otherwise

7. $f(x) = (\sin x)/x$.
Hint. Use Prob. 4 in Sec. 11.7.

8. Verify that u in the solution of Prob. 7 satisfies the initial condition.

9–12 CAS PROJECT. Error Function.

$$\text{erf}\, x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2}\, dw \tag{21}$$

This function is important in applied mathematics and physics (probability theory and statistics, thermodynamics, etc.) and fits our present discussion. Regarding it as a typical case of a special function defined by an integral that cannot be evaluated as in elementary calculus, do the following.

9. Graph the **bell-shaped curve** [the curve of the integrand in (21)]. Show that erf x is odd. Show that

$$\int_a^b e^{-w^2}\, dw = \frac{\sqrt{\pi}}{2}(\text{erf}\, b - \text{erf}\, a).$$

$$\int_{-b}^b e^{-w^2}\, dw = \sqrt{\pi}\, \text{erf}\, b.$$

10. Obtain the Maclaurin series of erf x from that of the integrand. Use that series to compute a table of erf x for $x = 0(0.01)3$ (meaning $x = 0, 0.01, 0.02, \cdots, 3$).

11. Obtain the values required in Prob. 10 by an integration command of your CAS. Compare accuracy.

12. It can be shown that erf $(\infty) = 1$. Confirm this experimentally by computing erf x for large x.

13. Let $f(x) = 1$ when $x > 0$ and 0 when $x < 0$. Using erf $(\infty) = 1$, show that (12) then gives

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2c\sqrt{t})}^{\infty} e^{-z^2} dz$$
$$= \frac{1}{2} - \frac{1}{2}\,\text{erf}\left(-\frac{x}{2c\sqrt{t}}\right) \qquad (t > 0).$$

14. Express the temperature (13) in terms of the error function.

15. Show that $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^x e^{-s^2/2}\, ds$
$$= \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x}{\sqrt{2}}\right).$$

Here, the integral is the definition of the "distribution function of the normal probability distribution" to be discussed in Sec. 24.8.

12.8 Modeling: Membrane, Two-Dimensional Wave Equation

Since the modeling here will be similar to that of Sec. 12.2, you may want to take another look at Sec. 12.2.

The vibrating string in Sec. 12.2 is a basic one-dimensional vibrational problem. Equally important is its two-dimensional analog, namely, the motion of an elastic membrane, such as a drumhead, that is stretched and then fixed along its edge. Indeed, setting up the model will proceed almost as in Sec. 12.2.

Physical Assumptions

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the xy-plane. The tension per unit length T caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Although these assumptions cannot be realized exactly, they hold relatively accurately for small transverse vibrations of a thin elastic membrane, so that we shall obtain a good model, for instance, of a drumhead.

Derivation of the PDE of the Model ("Two-Dimensional Wave Equation") from Forces. As in Sec. 12.2 the model will consist of a PDE and additional conditions. The PDE will be obtained by the same method as in Sec. 12.2, namely, by considering the forces acting on a small portion of the physical system, the membrane in Fig. 301 on the next page, as it is moving up and down.

Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to Δx and Δy. The tension T is the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\Delta x$ and $T\Delta y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

Horizontal Components of the Forces. We first consider the horizontal components of the forces. These components are obtained by multiplying the forces by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence the horizontal components of the forces at opposite sides are approximately equal. Therefore, the motion of the particles of the membrane in a horizontal direction will be negligibly small. From this we conclude that we may regard the motion of the membrane as transversal; that is, each particle moves vertically.

Vertical Components of the Forces. These components along the right side and the left side are (Fig. 301), respectively,

$$T\Delta y \sin\beta \quad \text{and} \quad -T\Delta y \sin\alpha.$$

Here α and β are the values of the angle of inclination (which varies slightly along the edges) in the middle of the edges, and the minus sign appears because the force on the that is,

$$w(x,t) = \sin\left(t - \frac{x}{c}\right) \quad \text{if} \quad \frac{x}{c} < t < \frac{x}{c} + 2\pi \quad \text{or} \quad ct > x > (t - 2\pi)c$$

and zero otherwise. This is a single sine wave traveling to the right with speed c. Note that a point x remains at rest until $t = x/c$, the time needed to reach that x if one starts at $t = 0$ (start of the motion of the left end) and travels with speed c. The result agrees with our physical intuition. Since we proceeded formally, we must verify that (5) satisfies the given conditions. We leave this to the student. ■

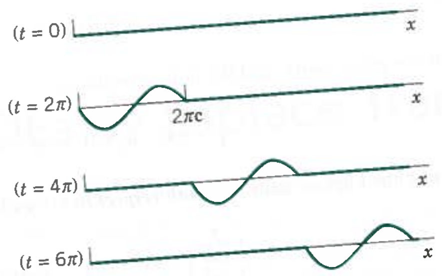

Fig. 317. Traveling wave in Example 1

We have reached the end of Chapter 12, in which we concentrated on the most important partial differential equations (PDEs) in physics and engineering. We have also reached the end of Part C on Fourier Analysis and PDEs.

Outlook

We have seen that PDEs underlie the modeling process of various important engineering application. Indeed, PDEs are the subject of many ongoing research projects.

Numerics for PDEs follows in Secs. 21.4–21.7, which, by design for greater flexibility in teaching, are independent of the other sections in Part E on numerics.

In the next part, that is, Part D on **complex analysis**, we turn to an area of a different nature that is also highly important to the engineer. The rich vein of examples and problems will signify this. It is of note that Part D includes another approach to the two-dimensional **Laplace equation** with applications, as shown in Chap. 18.

PROBLEM SET 12.12

1. Verify the solution in Example 1. What traveling wave do we obtain in Example 1 for a nonterminating sinusoidal motion of the left end starting at $t = 2\pi$?
2. Sketch a figure similar to Fig. 317 when $c = 1$ and $f(x)$ is "triangular," say, $f(x) = x$ if $0 < x < \frac{1}{2}$, $f(x) = 1 - x$ if $\frac{1}{2} < x < 1$ and 0 otherwise.
3. How does the speed of the wave in Example 1 of the text depend on the tension and on the mass of the string?

4–8 SOLVE BY LAPLACE TRANSFORMS

4. $\dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial t} = x, \quad w(x,0) = 1, \quad w(0,t) = 1$
5. $x\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial t} = xt, \quad w(x,0) = 0$ if $x \geqq 0$, $w(0,t) = 0$ if $t \geqq 0$
6. $\dfrac{\partial w}{\partial x} + 2x\dfrac{\partial w}{\partial t} = 2x, \quad w(x,0) = 1, \quad w(0,t) = 1$
7. Solve Prob. 5 by separating variables.
8. $\dfrac{\partial^2 w}{\partial x^2} = 100\dfrac{\partial^2 w}{\partial t^2} + 100\dfrac{\partial w}{\partial t} + 25w,$ $w(x,0) = 0$ if $x \geqq 0$, $w_t(x,0) = 0$ if $t \geqq 0$, $w(0,t) = \sin t$ if $t \geqq 0$

9–12 HEAT PROBLEM

Find the temperature $w(x,t)$ in a semi-infinite laterally insulated bar extending from $x = 0$ along the x-axis to infinity, assuming that the initial temperature is 0, $w(x,t) \to 0$ as $x \to \infty$ for every fixed $t \geqq 0$, and $w(0,t) = f(t)$. Proceed as follows.

9. Set up the model and show that the Laplace transform leads to
$$sW = c^2 \frac{\partial^2 W}{\partial x^2} \qquad (W = \mathcal{L}\{w\})$$
and
$$W = F(s)e^{-\sqrt{s}x/c} \qquad (F = \mathcal{L}\{f\}).$$
10. Applying the convolution theorem, show that in Prob. 9,
$$w(x,t) = \frac{x}{2c\sqrt{\pi}} \int_0^t f(t-\tau)\tau^{-3/2} e^{-x^2/(4c^2\tau)}\, d\tau.$$
11. Let $w(0,t) = f(t) = u(t)$ (Sec. 6.3). Denote the corresponding w, W, and F by w_0, W_0, and F_0. Show that then in Prob. 10,
$$w_0(x,t) = \frac{x}{2c\sqrt{\pi}} \int_0^t \tau^{-3/2} e^{-x^2/(4c^2\tau)}\, d\tau = 1 - \operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right)$$
with the error function erf as defined in Problem Set 12.7.
12. **Duhamel's formula.**[4] Show that in Prob. 11,
$$W_0(x,s) = \frac{1}{s} e^{-\sqrt{s}x/c}$$
and the convolution theorem gives *Duhamel's formula*
$$W(x,t) = \int_0^t f(t-\tau)\frac{\partial w_0}{\partial \tau}\, d\tau.$$

CHAPTER 12 REVIEW QUESTIONS AND PROBLEMS

1. For what kinds of problems will modeling lead to an ODE? To a PDE?
2. Mention some of the basic physical principles or laws that will give a PDE in modeling.
3. State three or four of the most important PDEs and their main applications.
4. What is "separating variables" in a PDE? When did we apply it twice in succession?
5. What is d'Alembert's solution method? To what PDE does it apply?
6. What role did Fourier series play in this chapter? Fourier integrals?
7. When and why did Legendre's equation occur? Bessel's equation?
8. What are the eigenfunctions and their frequencies of the vibrating string? Of the vibrating membrane?
9. What do you remember about types of PDEs? Normal forms? Why is this important?
10. When did we use polar coordinates? Cylindrical coordinates? Spherical coordinates?
11. Explain mathematically (not physically) why we got exponential functions in separating the heat equation, but not for the wave equation.
12. Why and where did the error function occur?
13. How do problems for the wave equation and the heat equation differ regarding additional conditions?
14. Name and explain the three kinds of boundary conditions for Laplace's equation.
15. Explain how the Laplace transform applies to PDEs.

16–18 Solve for $u = u(x, y)$:

16. $u_{xx} + 25u = 0$
17. $u_{yy} + u_y - 6u = 18$
18. $u_{xx} + u_x = 0, \quad u(0,y) = f(y), \quad u_x(0,y) = g(y)$

19–21 NORMAL FORM

Transform to normal form and solve:

19. $u_{xy} = u_{yy}$
20. $u_{xx} + 6u_{xy} + 9u_{yy} = 0$
21. $u_{xx} - 4u_{yy} = 0$

22–24 VIBRATING STRING

Find and sketch or graph (as in Fig. 288 in Sec. 12.3) the deflection $u(x,t)$ of a vibrating string of length π, extending from $x = 0$ to $x = \pi$, and $c^2 = T/\rho = 4$ starting with velocity zero and deflection:

22. $\sin 4x$
23. $\sin^3 x$
24. $\frac{1}{2}\pi - |x - \frac{1}{2}\pi|$

[4] JEAN-MARIE CONSTANT DUHAMEL (1797–1872), French mathematician.

EXAMPLE 5 Polynomials, Rational Functions

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function**. This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–8 REGIONS OF PRACTICAL INTEREST

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 2i| \leqq \frac{1}{4}$
2. $0 < |z| < 1$
3. $\frac{\pi}{2} < |z - 1 + 2i| < \pi$
4. $-\pi < \text{Im } z < \pi$
5. $|\arg z| < \frac{\pi}{3}$
6. $\text{Re}\,(1/z) < 1$
7. $\text{Re } z \leqq -1$
8. $|z + i| \geqq |z - i|$
9. **WRITING PROJECT. Sets in the Complex Plane.** Write a report by formulating the corresponding portions of the text in your own words and illustrating them with examples of your own.

COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 Function Values. Find Re f, and Im f and their values at the given point z.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$
11. $f(z) = 1/(1 + z)$ at $1 - i$
12. $f(z) = (z - 1)/(z + 1)$ at $2i$
13. **CAS PROJECT. Graphing Functions.** Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves Re $f(z)$ = const and Im $f(z)$ = const in the same figure, and the curves $|f(z)|$ = const in another figure, where **(a)** $f(z) = z^2$, **(b)** $f(z) = 1/z$, (c) $f(z) = z^4$.

14–17 Continuity. Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\text{Re } z^2)/|z|$
15. $|z|^2 \text{ Im }(1/z)$
16. $(\text{Im } z^2)/|z|^2$
17. $(\text{Re } z)/(1 - |z|)$

18–23 Differentiation. Find the value of the derivative of

18. $(z - i)/(z + i)$ at i
19. $(z - 2i)^3$ at $= 5 + 2i$
20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.
21. $i(1 - z)^n$ at 0
22. $(iz^3 + 3z^2)^3$ at $2i$
23. $z^3/(z - i)^3$ at $-i$
24. **TEAM PROJECT. Limit, Continuity, Derivative**

(a) Limit. Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re } f(z) = \text{Re } l, \qquad \lim_{z \to z_0} \text{Im } f(z) = \text{Im } l.$$

(b) Limit. If $\lim_{z \to z_0} f(x)$ exists, show that this limit is unique.

(c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

(d) Continuity. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

(e) Differentiability. Show that $f(z) = \text{Re } z = x$ is not differentiable at any z. Can you find other such functions?

(f) Differentiability. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

25. **WRITING PROJECT. Comparison with Calculus.** Summarize the second part of this section beginning with *Complex Function*, and indicate what is conceptually analogous to calculus and what is not.

13.4 Cauchy–Riemann Equations. Laplace's Equation

As we saw in the last section, to do complex analysis (i.e., "calculus in the complex") on any complex function, we require that function to be *analytic on some domain* that is differentiable in that domain.

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$(1) \qquad u_x = v_y, \qquad u_y = -v_x$$

everywhere in D; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for v) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z (see Example 3 in Sec. 13.3), and $u = x^2 - y^2$ and $v = 2xy$ satisfy (1), namely, $u_x = 2x = v_y$ as well as $u_y = -2y = -v_x$. More examples will follow.

THEOREM 1

Cauchy–Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) *at all points of D.*

[4] The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GenRef9] in App. 1.