For $N = 2^p$ this breakdown can be repeated $p - 1$ times in order to finally arrive at $N/2$ problems of size 2 each, so that the number of multiplications is reduced as indicated above.

We show the reduction from $N = 4$ to $M = N/2 = 2$ and then prove (22).

EXAMPLE 5 Fast Fourier Transform (FFT). Sample of $N = 4$ Values

When $N = 4$, then $w = w_N = -i$ as in Example 4 and $M = N/2 = 2$, hence $w = w_M = e^{-2\pi i/2} = e^{-\pi i} = -1$. Consequently,

$$\hat{\mathbf{f}}_{ev} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \end{bmatrix} = \mathbf{F}_2\mathbf{f}_{ev} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$\hat{\mathbf{f}}_{od} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \end{bmatrix} = \mathbf{F}_2\mathbf{f}_{od} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}.$$

From this and (22a) we obtain

$$\hat{f}_0 = \hat{f}_{ev,0} + w_N^0 \hat{f}_{od,0} = (f_0 + f_2) + (f_1 + f_3) = f_0 + f_1 + f_2 + f_3$$

$$\hat{f}_1 = \hat{f}_{ev,1} + w_N^1 \hat{f}_{od,1} = (f_0 - f_2) - i(f_1 + f_3) = f_0 - if_1 - f_2 + if_3.$$

Similarly, by (22b),

$$\hat{f}_2 = \hat{f}_{ev,0} - w_N^0 \hat{f}_{od,0} = (f_0 + f_2) - (f_1 + f_3) = f_0 - f_1 + f_2 - f_3$$

$$\hat{f}_3 = \hat{f}_{ev,1} - w_N^1 \hat{f}_{od,1} = (f_0 - f_2) - (-i)(f_1 - f_3) = f_0 + if_1 - f_2 - if_3.$$

This agrees with Example 4, as can be seen by replacing 0, 1, 4, 9 with f_0, f_1, f_2, f_3. ∎

We prove (22). From (18) and (19) we have for the components of the DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} w_N^{kn} f_k.$$

Splitting into two sums of $M = N/2$ terms each gives

$$\hat{f}_n = \sum_{k=0}^{M-1} w_N^{2kn} f_{2k} + \sum_{k=0}^{M-1} w_N^{(2k+1)n} f_{2k+1}.$$

We now use $w_N^2 = w_M$ and pull out w_N^n from under the second sum, obtaining

$$\hat{f}_n = \sum_{k=0}^{M-1} w_M^{kn} f_{ev,k} + w_N^n \sum_{k=0}^{M-1} w_M^{kn} f_{od,k}. \tag{23}$$

The two sums are $f_{ev,n}$ and $f_{od,n}$, the components of the "half-size" transforms $\mathbf{Ff}_{ev}$ and $\mathbf{Ff}_{od}$.

Formula (22a) is the same as (23). In (22b) we have $n + M$ instead of n. This causes a sign changes in (23), namely $-w_N^n$ before the second sum because

$$w_N^M = e^{-2\pi iM/N} = e^{-2\pi i/2} = e^{-\pi i} = -1.$$

This gives the minus in (22b) and completes the proof. ∎

PROBLEM SET 11.9

1. Review in complex. Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2\cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

2–11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

12–17 USE OF TABLE III IN SEC. 11.10. OTHER METHODS

12. Find $\mathscr{F}(f(x))$ for $f(x) = xe^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x}.

13. Obtain $\mathscr{F}(e^{-x^2/2})$ from Table III.

14. In Table III obtain formula 7 from formula 8.

15. In Table III obtain formula 1 from formula 2.

16. TEAM PROJECT. Shifting (a) Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathscr{F}\{f(x - a)\} = e^{-iwa}\mathscr{F}\{f(x)\}$.

(b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

(c) Shifting on the *w*-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

(d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.

17. What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

18–25 DISCRETE FOURIER TRANSFORM

18. Verify the calculations in Example 4 of the text.

19. Find the transform of a general signal $f = [f_1 \ \ f_2 \ \ f_3 \ \ f_4]^{\mathsf{T}}$ of four values.

20. Find the inverse matrix in Example 4 of the text and use it to recover the given signal.

21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \ \ f_2]^{\mathsf{T}}$.

22. Recreate the given signal in Prob. 21 from the frequency spectrum obtained.

23. Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.

24. Write the Fourier matrix $\mathbf{F}$ for a sample of eight values explicitly.

25. CAS Problem. Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.

Table III. Fourier Transforms

See (6) in Sec. 11.9.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|w\|}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi} w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
10	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}$ if $\|w\| < a$; 0 if $\|w\| > a$

CHAPTER 11 REVIEW QUESTIONS AND PROBLEMS

1. What is a Fourier series? A Fourier cosine series? A half-range expansion? Answer from memory.
2. What are the Euler formulas? By what very important idea did we obtain them?
3. How did we proceed from 2π-periodic to general-periodic functions?
4. Can a discontinuous function have a Fourier series? A Taylor series? Why are such functions of interest to the engineer?
5. What do you know about convergence of a Fourier series? About the Gibbs phenomenon?
6. The output of an ODE can oscillate several times as fast as the input. How come?
7. What is approximation by trigonometric polynomials? What is the minimum square error?
8. What is a Fourier integral? A Fourier sine integral? Give simple examples.
9. What is the Fourier transform? The discrete Fourier transform?
10. What are Sturm–Liouville problems? By what idea are they related to Fourier series?

11–20 FOURIER SERIES. In Probs. 11, 13, 16, 20 find the Fourier series of $f(x)$ as given over one period and sketch $f(x)$ and partial sums. In Probs. 12, 14, 15, 17–19 give answers, with reasons. Show your work detail.

11. $f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 2 \end{cases}$
12. Why does the series in Prob. 11 have no cosine terms?
13. $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$
14. What function does the series of the cosine terms in Prob. 13 represent? The series of the sine terms?
15. What function do the series of the cosine terms and the series of the sine terms in the Fourier series of e^x $(-5 < x < 5)$ represent?
16. $f(x) = |x| \quad (-\pi < x < \pi)$
17. Find a Fourier series from which you can conclude that $1 - 1/3 + 1/5 - 1/7 + - \cdots = \pi/4$.
18. What function and series do you obtain in Prob. 16 by (termwise) differentiation?
19. Find the half-range expansions of $f(x) = x$ $(0 < x < 1)$.
20. $f(x) = 3x^2 \quad (-\pi < x < \pi)$

21–22 GENERAL SOLUTION

Solve, $y'' + \omega^2 y = r(t)$, where $|\omega| \neq 0, 1, 2, \cdots, r(t)$ is 2π-periodic and

21. $r(t) = 3t^2 \ (-\pi < t < \pi)$
22. $r(t) = |t| \ (-\pi < t < \pi)$

23–25 MINIMUM SQUARE ERROR

23. Compute the minimum square error for $f(x) = x/\pi$ $(-\pi < x < \pi)$ and trigonometric polynomials of degree $N = 1, \cdots, 5$.
24. How does the minimum square error change if you multiply $f(x)$ by a constant k?
25. Same task as in Prob. 23, for $f(x) = |x|/\pi$ $(-\pi < x < \pi)$. Why is E^* now much smaller (by a factor 100, approximately!)?

26–30 FOURIER INTEGRALS AND TRANSFORMS

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing ∞ with finite limits; also look for Gibbs phenomena.

26. $f(x) = x + 1$ if $0 < x < 1$ and 0 otherwise; by the Fourier sine transform
27. $f(x) = x$ if $0 < x < 1$ and 0 otherwise; by the Fourier integral
28. $f(x) = kx$ if $a < x < b$ and 0 otherwise; by the Fourier transform
29. $f(x) = x$ if $1 < x < a$ and 0 otherwise; by the Fourier cosine transform
30. $f(x) = e^{-2x}$ if $x > 0$ and 0 otherwise; by the Fourier transform