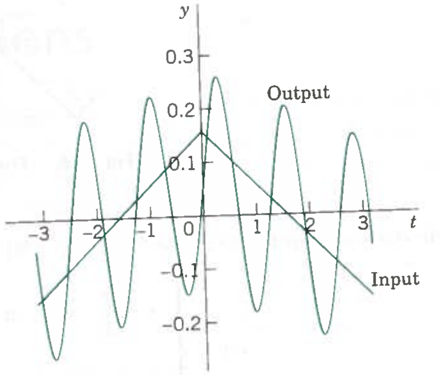

Fig. 277. Input and steady-state output in Example 1

PROBLEM SET 11.3

1. **Coefficients C_n.** Derive the formula for C_n from A_n and B_n.

2. **Change of spring and damping.** In Example 1, what happens to the amplitudes C_n if we take a stiffer spring, say, of $k = 49$? If we increase the damping?

3. **Phase shift.** Explain the role of the B_n's. What happens if we let $c \to 0$?

4. **Differentiation of input.** In Example 1, what happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?

5. **Sign of coefficients.** Some of the A_n in Example 1 are positive, some negative. All B_n are positive. Is this physically understandable?

6–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

6. $r(t) = \sin \alpha t + \sin \beta t, \omega^2 \neq \alpha^2, \beta^2$

7. $r(t) = \sin t, \omega = 0.5, 0.9, 1.1, 1.5, 10$

8. **Rectifier.** $r(t) = \pi/4\ |\cos t|$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t), |\omega| \neq 0, 2, 4, \cdots$

9. What kind of solution is excluded in Prob. 8 by $|\omega| \neq 0, 2, 4, \cdots$?

10. **Rectifier.** $r(t) = \pi/4\ |\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t), |\omega| \neq 0, 2, 4, \cdots$

11. $r(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ 1 & \text{if} \quad 0 < t < \pi, \end{cases} \quad |\omega| \neq 1, 3, 5, \cdots$

12. **CAS Program.** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 7 and 11 with initial values of your choice.

13–16 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$.

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$

14. $r(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ 1 & \text{if} \quad 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$

15. $r(t) = t(\pi^2 - t^2)$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$

16. $r(t) = \begin{cases} t & \text{if} \quad -\pi/2 < t < \pi/2 \\ \pi - t & \text{if} \quad \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

17–19 *RLC*-CIRCUIT

Find the steady-state current $I(t)$ in the *RLC*-circuit in Fig. 275, where $R = 10\ \Omega$, $L = 1$ H, $C = 10^{-1}$ F and with $E(t)$ V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. *Hint.* Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

17. $E(t) = \begin{cases} -50t^2 & \text{if} \quad -\pi < t < 0 \\ 50t^2 & \text{if} \quad 0 < t < \pi \end{cases}$

18. $E(t) = \begin{cases} 100\,(t - t^2) & \text{if} \quad -\pi < t < 0 \\ 100\,(t + t^2) & \text{if} \quad 0 < t < \pi \end{cases}$

19. $E(t) = 200t(\pi^2 - t^2) \quad (-\pi < t < \pi)$

20. **CAS EXPERIMENT. Maximum Output Term.** Graph and discuss outputs of $y'' + cy' + ky = r(t)$ with $r(t)$ as in Example 1 for various c and k with emphasis on the maximum C_n and its ratio to the second largest $|C_n|$.

11.4 Approximation by Trigonometric Polynomials

Fourier series play a prominent role not only in differential equations but also in **approximation theory**, an area that is concerned with approximating functions by other functions—usually simpler functions. Here is how Fourier series come into the picture.

Let $f(x)$ be a function on the interval $-\pi \leqq x \leqq \pi$ that can be represented on this interval by a Fourier series. Then the ***N*th partial sum** of the Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is an approximation of the given $f(x)$. In (1) we choose an arbitrary N and keep it fixed. Then we ask whether (1) is the "best" approximation of f by a **trigonometric polynomial of the same degree N**, that is, by a function of the form

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \qquad (N \text{ fixed}). \tag{2}$$

Here, "best" means that the "error" of the approximation is as small as possible.

Of course we must first define what we mean by the **error** of such an approximation. We could choose the maximum of $|f(x) - F(x)|$. But in connection with Fourier series it is better to choose a definition of error that measures the goodness of agreement between f and F *on the whole interval* $-\pi \leqq x \leqq \pi$. This is preferable since the sum f of a Fourier series may have jumps: F in Fig. 278 is a good overall approximation of f, but the maximum of $|f(x) - F(x)|$ (more precisely, the *supremum*) is large. We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2\, dx. \tag{3}$$

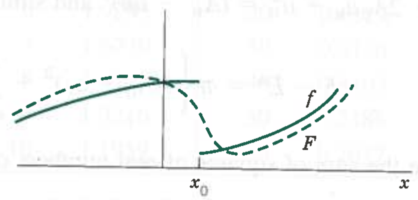

Fig. 278. Error of approximation

$F = S_1, S_2, S_3$ are shown in Fig. 269 in Sec. 11.2, and $F = S_{20}$ is shown in Fig. 279. Although $|f(x) - F(x)|$ is large at $\pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where "waves" remain owing to the "Gibbs phenomenon," which we shall discuss in the next section.

Can you think of functions f for which E^* decreases more quickly with increasing N? ■

ROBLEM SET 11.4

. **CAS Problem.** Do the numeric and graphic work in Example 1 in the text.

–5 MINIMUM SQUARE ERROR

nd the trigonometric polynomial $F(x)$ of the form (2) for hich the square error with respect to the given $f(x)$ on the erval $-\pi < x < \pi$ is minimum. Compute the minimum lue for $N = 1, 2, \cdots, 5$ (or also for larger values if you ve a CAS).

. $f(x) = x \quad (-\pi < x < \pi)$

3. $f(x) = |x| \quad (-\pi < x < \pi)$

4. $f(x) = x^2 \quad (-\pi < x < \pi)$

5. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$

6. Why are the square errors in Prob. 5 substantially larger than in Prob. 3?

7. $f(x) = x^3 \quad (-\pi < x < \pi)$

8. $f(x) = |\sin x| \quad (-\pi < x < \pi)$, full-wave rectifier

9. **Monotonicity.** Show that the minimum square error (6) is a monotone decreasing function of N. How can you use this in practice?

10. **CAS EXPERIMENT. Size and Decrease of E^*.** Compare the size of the minimum square error E^* for functions of your choice. Find experimentally the factors on which the decrease of E^* with N depends. For each function considered find the smallest N such that $E^* < 0.1$.

11–15 PARSEVALS'S IDENTITY

Using (8), prove that the series has the indicated sum. Compute the first few partial sums to see that the convergence is rapid.

11. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.233700550$

Use Example 1 in Sec. 11.1.

12. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = 1.082323234$

Use Prob. 14 in Sec. 11.1.

13. $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96} = 1.014678032$

Use Prob. 17 in Sec. 11.1.

14. $\int_{-\pi}^{\pi} \cos^4 x \, dx = \frac{3\pi}{4}$

15. $\int_{-\pi}^{\pi} \cos^6 x \, dx = \frac{5\pi}{8}$

11.5 Sturm–Liouville Problems. Orthogonal Functions

The idea of the Fourier series was to represent general periodic functions in terms of cosines and sines. The latter formed a *trigonometric system*. This trigonometric system has the desirable property of orthogonality which allows us to compute the coefficient of the Fourier series by the Euler formulas.

The question then arises, can this approach be generalized? That is, can we replace the trigonometric system of Sec. 11.1 by other *orthogonal systems* (*sets of other orthogonal functions*)? The answer is "yes" and will lead to generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series in Sec. 11.6.

To prepare for this generalization, we first have to introduce the concept of a Sturm–Liouville Problem. (The motivation for this approach will become clear as you read on.) Consider a second-order ODE of the form

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \tag{1}$$

on some interval $a \leqq x \leqq b$, satisfying conditions of the form

$$\text{(2)} \quad \begin{aligned} &\text{(a)} \quad k_1 y + k_2 y' = 0 \quad \text{at } x = a \\ &\text{(b)} \quad l_1 y + l_2 y' = 0 \quad \text{at } x = b. \end{aligned}$$

Here λ is a parameter, and k_1, k_2, l_1, l_2 are given real constants. Furthermore, at least one of each constant in each condition (2) must be different from zero. (We will see in Example 1 that, if $p(x) = r(x) = 1$ and $q(x) = 0$, then $\sin\sqrt{\lambda}x$ and $\cos\sqrt{\lambda}x$ satisfy (1) and constants can be found to satisfy (2).) Equation (1) is known as a **Sturm–Liouville equation.**[4] Together with conditions 2(a), 2(b) it is know as the **Sturm–Liouville problem.** It is an example of a boundary value problem.

A **boundary value problem** consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) $x = a$ and $x = b$ of a given interval $a \leqq x \leqq b$.

The goal is to solve these type of problems. To do so, we have to consider

Eigenvalues, Eigenfunctions

Clearly, $y \equiv 0$ is a solution—the **"trivial solution"**—of the problem (1), (2) for any λ because (1) is homogeneous and (2) has zeros on the right. This is of no interest. We want to find **eigenfunctions** $y(x)$, that is, solutions of (1) satisfying (2) without being identically zero. We call a number λ for which an eigenfunction exists an **eigenvalue** of the Sturm–Liouville problem (1), (2).

Many important ODEs in engineering can be written as Sturm–Liouville equations. The following example serves as a case in point.

EXAMPLE 1 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0. \tag{3}$$

This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and fixed at its ends $x = 0$ and $x = \pi$ and then allowed to vibrate. Then $y(x)$ is the "space function" of the deflection $u(x, t)$ of the string, assumed in the form $u(x, t) = y(x)w(t)$, where t is time. (This model will be discussed in great detail in Secs, 12.2–12.4.)

Solution. From (1) nad (2) we see that $p = 1, q = 0, r = 1$ in (1), and $a = 0, b = \pi, k_1 = l_1 = 1, k_2 = l_2 = 0$ in (2). For negative $\lambda = -\nu^2$ a general solution of the ODE in (3) is $y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$. From the boundary conditions we obtain $c_1 = c_2 = 0$, so that $y \equiv 0$, which is not an eigenfunction. For $\lambda = 0$ the situation is similar. For positive $\lambda = \nu^2$ a general solution is

$$y(x) = A\cos\nu x + B\sin\nu x.$$

[4]JACQUES CHARLES FRANÇOIS STURM (1803–1855) was born and studied in Switzerland and then moved to Paris, where he later became the successor of Poisson in the chair of mechanics at the Sorbonne (the University of Paris).

JOSEPH LIOUVILLE (1809–1882), French mathematician and professor in Paris, contributed to various fields in mathematics and is particularly known by his important work in complex analysis (Liouville's theorem; Sec. 14.4), special functions, differential geometry, and number theory.

Table III. Fourier Transforms

See (6) in Sec. 11.9.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\,\frac{e^{-a\lvert w\rvert}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}\,w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{i}{\sqrt{2\pi}}\,\frac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}}\,e^{-w^2/4a}$
10	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}$ if $\lvert w\rvert < a$; 0 if $\lvert w\rvert > a$

CHAPTER 11 REVIEW QUESTIONS AND PROBLEMS

1. What is a Fourier series? A Fourier cosine series? A half-range expansion? Answer from memory.
2. What are the Euler formulas? By what very important idea did we obtain them?
3. How did we proceed from 2π-periodic to general-periodic functions?
4. Can a discontinuous function have a Fourier series? A Taylor series? Why are such functions of interest to the engineer?
5. What do you know about convergence of a Fourier series? About the Gibbs phenomenon?
6. The output of an ODE can oscillate several times as fast as the input. How come?
7. What is approximation by trigonometric polynomials? What is the minimum square error?
8. What is a Fourier integral? A Fourier sine integral? Give simple examples.
9. What is the Fourier transform? The discrete Fourier transform?
10. What are Sturm–Liouville problems? By what idea are they related to Fourier series?

11–20 FOURIER SERIES. In Probs. 11, 13, 16, 20 find the Fourier series of $f(x)$ as given over one period and sketch $f(x)$ and partial sums. In Probs. 12, 14, 15, 17–19 give answers, with reasons. Show your work detail.

11. $f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 2 \end{cases}$
12. Why does the series in Prob. 11 have no cosine terms?
13. $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$
14. What function does the series of the cosine terms in Prob. 13 represent? The series of the sine terms?
15. What function do the series of the cosine terms and the series of the sine terms in the Fourier series of e^x $(-5 < x < 5)$ represent?
16. $f(x) = \lvert x\rvert \quad (-\pi < x < \pi)$
17. Find a Fourier series from which you can conclude that $1 - 1/3 + 1/5 - 1/7 + - \cdots = \pi/4$.
18. What function and series do you obtain in Prob. 16 by (termwise) differentiation?
19. Find the half-range expansions of $f(x) = x$ $(0 < x < 1)$.
20. $f(x) = 3x^2 \quad (-\pi < x < \pi)$

21–22 GENERAL SOLUTION

Solve, $y'' + \omega^2 y = r(t)$, where $\lvert\omega\rvert \neq 0, 1, 2, \cdots, r(t)$ is 2π-periodic and

21. $r(t) = 3t^2 \ (-\pi < t < \pi)$
22. $r(t) = \lvert t\rvert \ (-\pi < t < \pi)$

23–25 MINIMUM SQUARE ERROR

23. Compute the minimum square error for $f(x) = x/\pi$ $(-\pi < x < \pi)$ and trigonometric polynomials of degree $N = 1, \cdots, 5$.
24. How does the minimum square error change if you multiply $f(x)$ by a constant k?
25. Same task as in Prob. 23, for $f(x) = \lvert x\rvert/\pi$ $(-\pi < x < \pi)$. Why is E^* now much smaller (by a factor 100, approximately!)?

26–30 FOURIER INTEGRALS AND TRANSFORMS

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing ∞ with finite limits; also look for Gibbs phenomena.

26. $f(x) = x + 1$ if $0 < x < 1$ and 0 otherwise; by the Fourier sine transform
27. $f(x) = x$ if $0 < x < 1$ and 0 otherwise; by the Fourier integral
28. $f(x) = kx$ if $a < x < b$ and 0 otherwise; by the Fourier transform
29. $f(x) = x$ if $1 < x < a$ and 0 otherwise; by the Fourier cosine transform
30. $f(x) = e^{-2x}$ if $x > 0$ and 0 otherwise; by the Fourier transform