11.4 Approximation by Trigonometric Polynomials

Fourier series play a prominent role not only in differential equations but also in approximation theory, an area that is concerned with approximating functions by other functions—usually simpler functions. Here is how Fourier series come into the picture.

Let \( f(x) \) be a function on the interval \( -\pi \leq x \leq \pi \) that can be represented on this interval by a Fourier series. Then the \( N \)th partial sum of the Fourier series

\[
f(x) = a_0 + \sum_{n=1}^{N} \left( a_n \cos nx + b_n \sin nx \right)
\]

is an approximation of the given \( f(x) \). In (1) we choose an arbitrary \( N \) and keep it fixed. Then we ask whether (1) is the “best” approximation of \( f \) by a trigonometric polynomial of the same degree \( N \), that is, by a function of the form

\[
F(x) = A_0 + \sum_{n=1}^{N} \left( A_n \cos nx + B_n \sin nx \right)
\]

(2) \( N \) fixed.

Here, “best” means that the “error” of the approximation is as small as possible.

Of course we must first define what we mean by the error of such an approximation. We could choose the maximum of \( |f(x) - F(x)| \). But in connection with Fourier series it is better to choose a definition of error that measures the goodness of agreement between \( f \) and \( F \) on the whole interval \( -\pi \leq x \leq \pi \). This is preferable since the sum of a Fourier series may have jumps; \( F \) in Fig. 278 is a good overall approximation of \( f \), but the maximum of \( |f(x) - F(x)| \) (more precisely, the supremum) is large. We choose

\[
E = \int_{-\pi}^{\pi} (f - F)^2 \, dx.
\]

SEC. 11.4 Approximation by Trigonometric Polynomials

18. \( E(t) = \begin{cases} 100(t - t^2) & \text{if } -\pi < t < 0 \\ 100(t + t^2) & \text{if } 0 < t < \pi \end{cases} \)
20. CAS EXPERIMENT. Maximum Output Term. Graph and discuss outputs of \( y'' + cy' + ky = r(t) \) with \( r(t) \) as in Example 1 for various \( c \) and \( k \) with emphasis on the maximum of \( C_n \) and its ratio to the second largest \( |C_n| \).

11.4 Approximation by Trigonometric Polynomials

1. Coefficients \( C_n \). Derive the formula for \( C_n \) from \( A_n \) and \( B_n \).
2. Change of spring and damping. In Example 1, what happens to the amplitudes \( C_n \) if we take a stiffer spring, say, of \( k = 49 \)? If we increase the damping?
3. Phase shift. Explain the role of the \( B_n \)‘s. What happens if we let \( c = 0 \)?
4. Differentiation of input. In Example 1, what happens if we replace \( r(t) \) with its derivative, the rectangular wave?
5. Sign of coefficients. Some of the \( A_n \) in Example 1 are positive, some are negative. All \( B_n \) are positive. Is this physically understandable?

6-11 GENERAL SOLUTION

13. Find a general solution of the ODE \( y'' + \omega^2 y = r(t) \) with \( r(t) \) as given. Show the details of your work.
14. \( r(t) = \sin nt \), \( t = 0.5, 0.9, 1.1, 1.5, 10 \)
15. Rectifier. \( r(t) = \arcsin t \), \( \pi/2 < t < \pi \) and \( r(t) = r(t), |r(t)| = 0, 2, 4, \ldots \)
16. What kind of solution is excluded in Prob. 8 by \( |r(t)| = 0, 2, 4, \ldots \)?
17. Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 7 and 11 with initial values of your choice.
11.5 Sturm–Liouville Problems. Orthogonal Functions.

The idea of the Fourier series was to represent general periodic functions in terms of cosines and sines. The latter formed a trigonometric system. This trigonometric system has the desirable property of orthonormality which allows us to compute the coefficient of the Fourier series by the Euler formulas.

The question then arises, can this approach be generalized? That is, can we replace the trigonometric system of Sec. 11.1 by other orthogonal systems (sets of other orthogonal functions)? The answer is "yes" and will lead to generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series in Sec. 11.6.

To prepare for this generalization, we first have to introduce the concept of a Sturm–Liouville Problem. (The motivation for this approach will become clear as you read on.)

Consider a second-order ODE of the form

\[ (p(x)y')' + (q(x) + \lambda r(x))y = 0 \]

on some interval \( a \leq x \leq b \), satisfying conditions of the form

(a) \( ky_1 + k_2 y_2 = 0 \) at \( x = a \)

(b) \( l_1 y_1 + l_2 y_2 = 0 \) at \( x = b \).

Here \( A \) is a parameter, and \( k_1, k_2, l_1, l_2 \) are given real constants. Furthermore, at least one of each constant in condition (2) must be different from zero. We will see in Example 1 that, if \( p(x) = x^2 \) and \( q(x) = 0 \), then \( \sin \sqrt{x} \) and \( \cos \sqrt{x} \) satisfy (1) and (2) and can be found to satisfy (2). Equation (1) is known as a Sturm–Liouville equation. Together with conditions 2(a), 2(b), it is known as the Sturm–Liouville problem. It is an example of a boundary value problem.

A boundary value problem consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) \( x = a \) and \( x = b \) of a given interval \( a \leq x \leq b \).

The goal is to solve these type of problems. To do so, we have to consider

**Eigenvalues, Eigenfunctions.**

Clearly, \( y = 0 \) is a solution—the "trivial solution"—of the problem (1), (2) for any \( \lambda \) because (1) is homogeneous and (2) has zeros on the right. This is of no interest. We want to find eigenfunctions \( y(x) \), that is, solutions of (1) satisfying (2) without being identically zero. We call a number \( \lambda \) for which an eigenfunction exists an eigenvalue of the Sturm–Liouville problem (1), (2).

Many important ODEs in engineering can be written as Sturm–Liouville equations. The following example serves as a case in point.

**Example 1.** Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

\[ \frac{d}{dx} \left( \frac{d}{dx} y(x) \right) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(\pi) = 0, \]

This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and fixed at its ends \( x = 0 \) and \( x = \pi \) and then allowed to vibrate. Then \( y(x) \) is the "space function" of the deflection \( u(x, t) \) of the string, assumed in the form \( u(x, t) = y(x) \phi(t) \), where \( t \) is time. (This model will be discussed in great detail in Secs. 12.2–12.4.)

**Solution.** From (1) and (2) we see that \( p = 1, q = 0, \) so \( \lambda = \beta^2 \) in (1), and \( \lambda = 0, \beta = \pi, k_1 = 1, k_2 = 0 \) in (2). For negative \( \lambda = -\pi^2 \) a general solution of the ODE in (3) is \( y(x) = c_1 \cos \pi x + c_2 \sin \pi x \). From the boundary conditions we obtain \( c_1 = c_2 = 0 \), so that \( y = 0 \), which is not an eigenfunction. For positive \( \lambda = \pi^2 \) a general solution is

\[ y(x) = \phi_0 \cos \pi x + \phi_1 \sin \pi x. \]
### Table III. Fourier Transforms

See (6) in Sec. 11.9.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f(w) = \mathcal{F}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{cases} 1 &amp; \text{if } -b &lt; x &lt; b \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{2}{\pi} \frac{\sin bw}{w}$</td>
</tr>
<tr>
<td>$\begin{cases} 1 &amp; \text{if } b &lt; x &lt; c \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{e^{-iw(b-c)} - e^{-iw(a)}}{iw\sqrt{2\pi}}$</td>
</tr>
<tr>
<td>$\frac{1}{x^2 + a^2}$ $(a &gt; 0)$</td>
<td>$\frac{\pi e^{-awi}}{2a}$</td>
</tr>
<tr>
<td>$\begin{cases} x &amp; \text{if } 0 &lt; x &lt; b \ 2x - b &amp; \text{if } b &lt; x &lt; 2b \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{-1 + 2e^{iwb} - e^{-iwb}}{\sqrt{2\pi w^3}}$</td>
</tr>
<tr>
<td>$e^{-ax}$ $(a &gt; 0)$</td>
<td>$\begin{cases} 1 &amp; \text{if } x &gt; 0 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>$\begin{cases} e^{iax} &amp; \text{if } b &lt; x &lt; c \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{e^{i(a-w)c} - e^{i(a+b)x}}{\sqrt{2\pi}(a - iw)}$</td>
</tr>
<tr>
<td>$\begin{cases} i &amp; \text{if } -b &lt; x &lt; b \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{2}{\pi} \frac{\sin b(w-a)}{w-a}$</td>
</tr>
<tr>
<td>$\begin{cases} i &amp; \text{if } b &lt; x &lt; c \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{i}{\sqrt{2\pi}} \frac{e^{i(a-w)c} - e^{i(a-w)x}}{a-w}$</td>
</tr>
<tr>
<td>$e^{-ax^2}$ $(a &gt; 0)$</td>
<td>$\frac{1}{\sqrt{2\pi a}} e^{-a</td>
</tr>
<tr>
<td>$\sin \frac{ax}{a}$ $(a &gt; 0)$</td>
<td>$\sqrt{\frac{\pi}{2}}$ if $</td>
</tr>
</tbody>
</table>

**Chapter 11 Review Questions and Problems**

2. What are the Euler formulas? By what very important idea did we obtain them?
3. How did we proceed from $2\pi$-periodic to general-periodic functions?
4. Can a discontinuous function have a Fourier series? A Taylor series? Why are such functions of interest to the engineer?
5. What do you know about convergence of a Fourier series? About the Gibbs phenomenon?
6. The output of an O.D.E. can oscillate several times as fast as the input. How come?
7. What is approximation by trigonometric polynomials? What is the minimum square error?
9. What is the Fourier transform? The discrete Fourier transform?
10. What are Sturm–Liouville problems? By what idea are they related to Fourier series?

**FOURIER SERIES.** In Probs. 11, 13, 16, 20 find the Fourier series of $f(x)$ as given over one period and sketch $f(x)$ and partial sums. In Probs. 12, 14, 15, 17–19 give answers, with reasons. Show your work detail.

11. $f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 1 & \text{if } 0 < x < 2 \end{cases}$
12. Why does the series in Prob. 11 have no cosine terms?
13. $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}$
14. What function does the series of the cosine terms in Prob. 13 represent? The series of the sine terms?
15. What function do the series of the cosine terms and the series of the sine terms in the Fourier series of $e^{i}(-5 < x < 5)$ represent?
16. $f(x) = |x|$ $(-\pi < x < \pi)$
17. Find a Fourier series from which you can conclude that $1 - \frac{1}{15} + \frac{1}{17} - \frac{1}{21} + \cdots = \pi/4$.
18. What function and series do you obtain in Prob. 16 by (termwise) differentiation?
19. Find the half-range expansions of $f(x) = x$ $(0 < x < 1)$.
20. $f(x) = 3\delta$ $(-\pi < x < \pi)$

**GENERAL SOLUTION**

21–22

21. $r(t) = 3\delta(-\pi < t < \pi)$
22. $r(t) = \delta(-\pi < t < \pi)$

**MINIMUM SQUARE ERROR**

23. Compute the minimum square error for $f(x) = x/\pi$ $(-\pi < x < \pi)$ and trigonometric polynomials of degree $N = 1, \ldots, 5$.
24. How does the minimum square error change if you multiply $f(x)$ by a constant $k$?
25. Same task as in Prob. 23, for $f(x) = |x|/\pi$ $(-\pi < x < \pi)$. Why is $E_n$ now much smaller (by a factor 100, approximately)?

**FOURIER INTEGRALS AND TRANSFORMS**

26–30

26. $f(x) = x + 1$ if $0 < x < 1$ and 0 otherwise; by the Fourier sine transform
27. $f(x) = 1$ if $0 < x < 1$ and 0 otherwise; by the Fourier integral
28. $f(x) = \cos x$ if $-\pi < x < \pi$ and 0 otherwise; by the Fourier transform
29. $f(x) = x$ if $0 < x < 1$ and 0 otherwise; by the Fourier cosine transform
30. $f(x) = e^{-2x}$ if $x > 0$ and 0 otherwise; by the Fourier transform