PROBLEM SET 11.1

1–5 PERIOD, FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$
2. $\cos nx$, $\sin nx$, $\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$, $\cos \frac{2\pi nx}{k}$, $\sin \frac{2\pi nx}{k}$
3. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ (a, b, constant) has the period p. Thus all functions of period p form a **vector space**.
4. **Change of scale.** If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp, respectively. Give examples.
5. Show that $f =$ const is periodic with any period but has no fundamental period.

6–10 GRAPHS OF 2π–PERIODIC FUNCTIONS

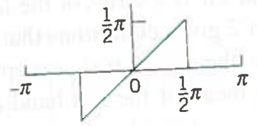
Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$
7. $f(x) = |\sin x|$, $f(x) = \sin |x|$
8. $f(x) = e^{-|x|}$, $f(x) = |e^{-x}|$
9. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$
10. $f(x) = \begin{cases} -\cos^2 x & \text{if } -\pi < x < 0 \\ \cos^2 x & \text{if } 0 < x < \pi \end{cases}$
11. **Calculus review.** Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

12–21 FOURIER SERIES

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

12. $f(x)$ in Prob. 6
13. $f(x)$ in Prob. 9
14. $f(x) = x^2$ $(-\pi < x < \pi)$
15. $f(x) = x^2$ $(0 < x < 2\pi)$
16.



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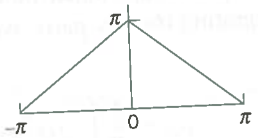

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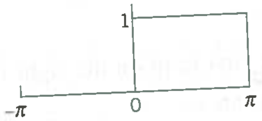

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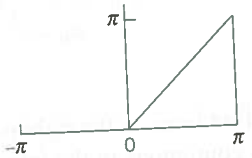

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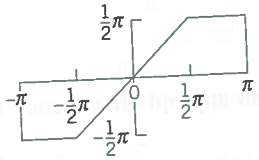

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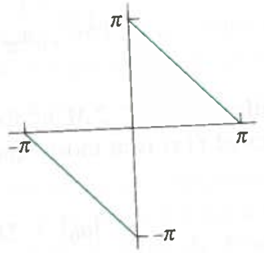


22. **CAS EXPERIMENT. Graphing.** Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.
   (a) $2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$
   $\quad - 2(\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x \cdots)$
   (b) $\frac{1}{2} + \frac{4}{\pi^2}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$
   (c) $\frac{2}{3}\pi^2 + 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots)$
23. **Discontinuities.** Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 21.
24. **CAS EXPERIMENT. Orthogonality.** Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos mx$ and $\cos nx$ $(m \neq n)$ for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2, \pi/3, \pi/4$? Other a? Extend the experiment to $\cos mx \sin nx$ and $\sin mx \sin nx$.
25. **CAS EXPERIMENT. Order of Fourier Coefficients.** The order seems to be $1/n$ if f is discontinous, and $1/n^2$ if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

We now expand our initial basic discussion of Fourier series.

Orientation. This section concerns three topics:

1. Transition from period 2π to any period $2L$, for the function f, simply by a transformation of scale on the x-axis.
2. Simplifications. Only cosine terms if f is even ("Fourier cosine series"). Only sine terms if f is odd ("Fourier sine series").
3. Expansion of f given for $0 \leqq x \leqq L$ in two Fourier series, one having only cosine terms and the other only sine terms ("half-range expansions").

1. From Period 2π to Any Period $p = 2L$

Clearly, periodic functions in applications may have any period, not just 2π as in the last section (chosen to have simple formulas). The notation $p = 2L$ for the period is practical because L will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.

The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable v such that $f(x)$, as a function of v, has period 2π. If we set

$$(1) \qquad \text{(a)} \quad x = \frac{p}{2\pi} v, \qquad \text{so that} \qquad \text{(b)} \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x$$

then $v = \pm\pi$ corresponds to $x = \pm L$. This means that f, as a function of v, has period 2π and, therefore, a Fourier series of the form

$$(2) \qquad f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with coefficients obtained from (6) in the last section

$$(3) \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos nv \, dv,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin nv \, dv.$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (6**) we obtain

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5. ∎

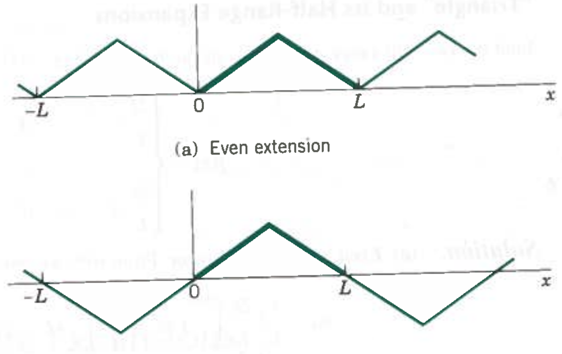


Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

1–7 EVEN AND ODD FUNCTIONS

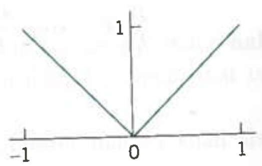
Are the following functions even or odd or neither even nor odd?

1. e^x, $e^{-|x|}$, $x^3\cos nx$, $x^2\tan\pi x$, $\sinh x - \cosh x$
2. $\sin^2 x$, $\sin(x^2)$, $\ln x$, $x/(x^2+1)$, $x\cot x$
3. Sums and products of even functions
4. Sums and products of odd functions
5. Absolute values of odd functions
6. Product of an odd times an even function
7. Find all functions that are both even and odd.

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

8.



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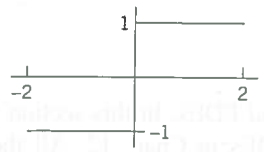

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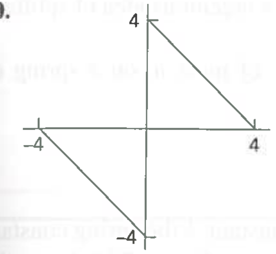
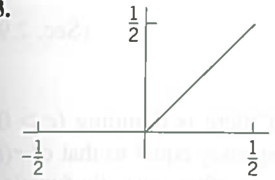
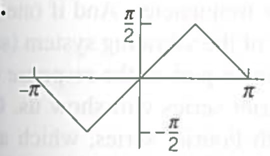


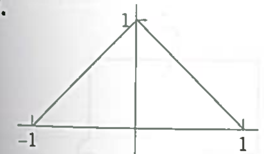
11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$
13.



14. $f(x) = \cos\pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$
15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$
17.



18. **Rectifier.** Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0\cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.
19. **Trigonometric Identities.** Show that the familiar identities $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ and $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.
20. **Numeric Values.** Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.
21. **CAS PROJECT. Fourier Series of 2L-Periodic Functions.** (a) Write a program for obtaining partial sums of a Fourier series (5).

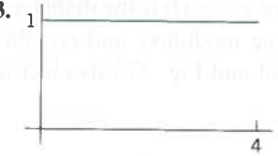
(b) Apply the program to Probs. 8–11, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.

22. Obtain the Fourier series in Prob. 8 from that in Prob. 17.

23–29 HALF-RANGE EXPANSIONS

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

23.



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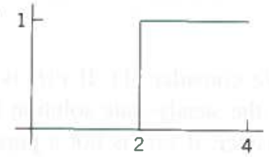

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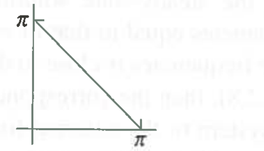

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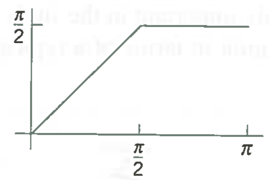

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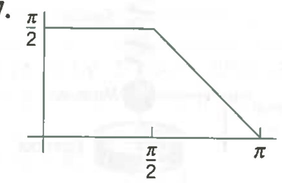

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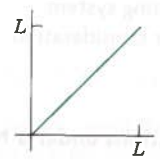

29. $f(x) = \sin x \; (0 < x < \pi)$
30. Obtain the solution to Prob. 26 from that of Prob. 27.

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 322 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

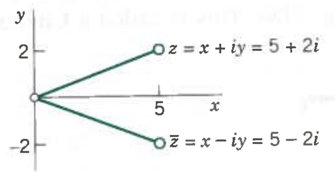

Fig. 322. Complex conjugate numbers

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

(8) $$\text{Re } z = x = \tfrac{1}{2}(z + \bar{z}), \qquad \text{Im } z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

(9) $$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\text{Im } z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$
$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$ ∎

PROBLEM SET 13.1

1. **Powers of i.** Show that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \cdots$ and $1/i = -i, 1/i^2 = -1, 1/i^3 = i, \cdots$.
2. **Rotation.** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 1 + i, z = -1 + 2i, z = 4 - 3i$.
3. **Division.** Verify the calculation in (7). Apply (7) to $(26 - 18i)/(6 - 2i)$.

4. **Law for conjugates.** Verify (9) for $z_1 = -11 + 10i$, $z_2 = -1 + 4i$.
5. **Pure imaginary number.** Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **Multiplication.** If the product of two complex numbers is zero, show that at least one factor must be zero.
7. **Laws of addition and multiplication.** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \; z_1 z_2 = z_2 z_1 \quad (\textit{Commutative laws})$$
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2)z_3 = z_1(z_2 z_3) \quad (\textit{Associative laws})$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad (\textit{Distributive law})$$
$$0 + z = z + 0 = z,$$
$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

8–15 COMPLEX ARITHMETIC

Let $z_1 = -2 + 5i$, $z_2 = 3 - i$. Showing the details of your work, find, in the form $x + iy$:

8. $z_1 z_2$, $\overline{(z_1 z_2)}$
9. $\text{Re}(z_1^2)$, $(\text{Re } z_1)^2$
10. $\text{Re}(1/z_2^2)$, $1/\text{Re}(z_2^2)$
11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$
12. z_1/z_2, z_2/z_1
13. $(z_1 + z_2)(z_1 - z_2)$, $z_1^2 - z_2^2$
14. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
15. $4(z_1 + z_2)/(z_1 - z_2)$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}(1/z)$, $\text{Im}(1/z^2)$
17. $\text{Re } z^4 - (\text{Re } z^2)^2$
18. $\text{Re}[(1 + i)^{16} z^2]$
19. $\text{Re}(z/\bar{z})$, $\text{Im}(z/\bar{z})$
20. $\text{Im}(1/\bar{z}^2)$

13.2 Polar Form of Complex Numbers. Powers and Roots

We gain further insight into the arithmetic operations of complex numbers if, in addition to the xy-coordinates in the complex plane, we also employ the usual polar coordinates r, θ defined by

(1) $$x = r\cos\theta, \qquad y = r\sin\theta.$$

We see that then $z = x + iy$ takes the so-called **polar form**

(2) $$z = r(\cos\theta + i\sin\theta).$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

(3) $$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 323). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 324).

θ is called the **argument** of z and is denoted by arg z. Thus $\theta = \arg z$ and (Fig. 323)

(4) $$\tan\theta = \frac{y}{x} \qquad (z \neq 0).$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 323. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense.***

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number z ($\neq 0$), then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \quad w_1\omega, \quad w_1\omega^2, \quad \cdots, \quad w_1\omega^{n-1} \tag{17}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

PROBLEM SET 13.2

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$
2. $-2 + 2i$
3. $2i, \quad -2i$
4. -4
5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$
6. $\dfrac{\sqrt{5} - 10i}{-\frac{1}{2}\sqrt{5} + 5i}$
7. $1 + \frac{1}{2}\pi i$
8. $\dfrac{7 + 4i}{3 - 2i}$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $1 - i$
10. $-5, \quad -5 - i, \quad -5 + i$
11. $\sqrt{3} \pm i$
12. $-\pi - \pi i$
13. $(1 - i)^{20}$
14. $-1 + 0.1i, \quad -1 - 0.1i$

15–18 CONVERSION TO $x + iy$

Graph in the complex plane and represent in the form $x + iy$:

15. $4(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})$
16. $6(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$
17. $\sqrt{8}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$
18. $\sqrt{50}(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$

ROOTS

19. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

20. **TEAM PROJECT. Square Root. (a)** Show that $w = \sqrt{z}$ has the values

$$
\begin{aligned}
w_1 &= \sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right], \\
w_2 &= \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right] \\
&= -w_1.
\end{aligned} \tag{18}
$$

(b) Obtain from (18) the often more practical formula

$$\sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y)i\sqrt{\tfrac{1}{2}(|z| - x)}\right] \tag{19}$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $-14i$, $-9 - 40i$, and $1 + \sqrt{48}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

21–27 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt[3]{1 - i}$
22. $\sqrt[3]{3 + 4i}$
23. $\sqrt[3]{343}$
24. $\sqrt[4]{-4}$
25. $\sqrt[4]{i}$
26. $\sqrt[8]{1}$
27. $\sqrt[5]{-1}$

28–31 EQUATIONS

Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$
29. $z^2 - z + 1 + i = 0$
30. $z^4 + 324 = 0$. Using the solutions, factor $z^4 + 324$ into quadratic factors with *real* coefficients.
31. $z^4 - 6iz^2 + 16 = 0$

32–35 INEQUALITIES AND EQUALITY

32. **Triangle inequality.** Verify (6) for $z_1 = 3 + i$, $z_2 = -2 + 4i$
33. **Triangle inequality.** Prove (6).
34. **Re and Im.** Prove $|\text{Re}\, z| \leqq |z|$, $\quad |\text{Im}\, z| \leqq |z|$.
35. **Parallelogram equality.** Prove and explain the name

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

13.3 Derivative. Analytic Function

Just as the study of calculus or real analysis required concepts such as domain, neighborhood, function, limit, continuity, derivative, etc., so does the study of complex analysis. Since the functions live in the complex plane, the concepts are slightly more difficult or *different* from those in real analysis. This section can be seen as a reference section where many of the concepts needed for the rest of Part D are introduced.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

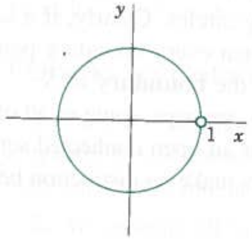

Fig. 330. Unit circle

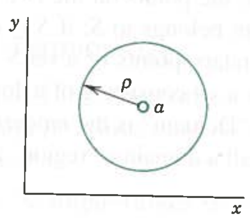

Fig. 331. Circle in the complex plane

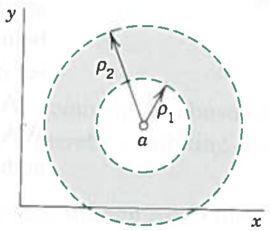

Fig. 332. Annulus in the complex plane

because it is the set of all z whose distance $|z - a|$ from the center a equals ρ. Accordingly, its interior (**"open circular disk"**) is given by $|z - a| < \rho$, its interior plus the circle itself (**"closed circular disk"**) by $|z - a| \leqq \rho$, and its exterior by $|z - a| > \rho$. As an example, sketch this for $a = 1 + i$ and $\rho = 2$, to make sure that you understand these inequalities.

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of a or, more precisely, a *ρ-neighborhood* of a. And a has infinitely many of them, one for each value of ρ (> 0), and a is a point of each of them, by definition!

In modern literature *any set* containing a ρ-neighborhood of a is also called a *neighborhood of a*.

Figure 332 shows an **open annulus** (circular ring) $\rho_1 < |z - a| < \rho_2$, which we shall need later. This is the set of all z whose distance $|z - a|$ from a is greater than ρ_1 but less than ρ_2. Similarly, the **closed annulus** $\rho_1 \leqq |z - a| \leqq \rho_2$ includes the two circles.

Half-Planes. By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.