The sum of this inverse and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$y(t) = 3e^{-t} \cos t - 2 \cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi. \tag{9}$$

In the second fraction in (6), taken with the minus sign, we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform of this fraction for $t > 0$ in the form

$$+2 \cos (2t - 2\pi) + \sin (2t - 2\pi) - e^{-(t-\pi)}[2 \cos (t - \pi) + 4 \sin (t - \pi)]$$

$$= 2 \cos 2t + \sin 2t + e^{-(t-\pi)} (2 \cos t + 4 \sin t).$$

The sum of this and (9) is the solution for $t > \pi$,

$$y(t) = e^{-t}[(3 + 2e^{\pi}) \cos t + 4e^{\pi} \sin t] \qquad \text{if } t > \pi. \tag{10}$$

Figure 136 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■

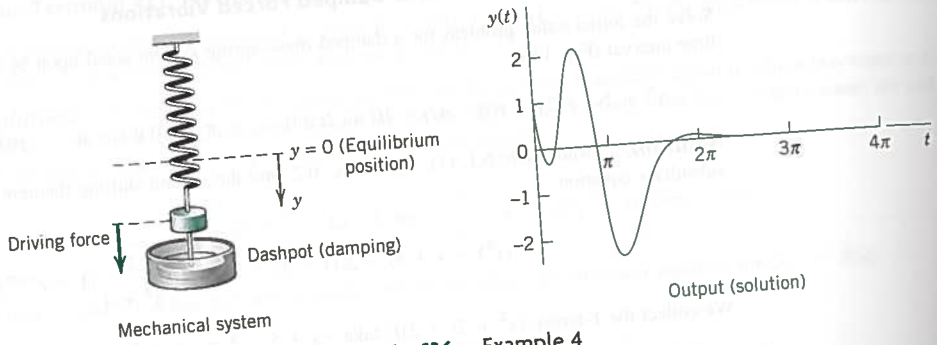

Fig. 136. Example 4

The case of repeated complex factors $[(s - a)(s - \bar{a})]^2$, which is important in connection with resonance, will be handled by "convolution" in the next section.

PROBLEM SET 6.4

1. CAS PROJECT. Effect of Damping. Consider a vibrating system of your choice modeled by

$$y'' + cy' + ky = \delta(t).$$

(a) Using graphs of the solution, describe the effect of continuously decreasing the damping to 0, keeping k constant.

(b) What happens if c is kept constant and k is continuously increased, starting from 0?

(c) Extend your results to a system with two δ-functions on the right, acting at different times.

2. CAS EXPERIMENT. Limit of a Rectangular Wave. Effects of Impulse.

(a) In Example 1 in the text, take a rectangular wave of area 1 from 1 to $1 + k$. Graph the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 134. *Hint:* If your CAS gives no solution for the differential equation, involving k, take specific k's from the beginning.

(b) Experiment on the response of the ODE in Example 1 (or of another ODE of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting graphs?

3–12 EFFECT OF DELTA (IMPULSE) ON VIBRATING SYSTEMS

Find and graph or sketch the solution of the IVP. Show the details.

3. $y'' + 9y = \delta(t - \pi/2), \quad y(0) = 2, y'(0) = 0$

4. $y'' + 16y = 4\delta(t - 3\pi), \quad y(0) = 2, y'(0) = 0$

5. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi),$
$y(0) = 0, y'(0) = 1$

6. $y'' + 4y' + 5y = \delta(t - 1), \quad y(0) = 0, y'(0) = 3$

7. $4y'' + 16y' + 17y = 3e^{-t} + \delta(t - \frac{1}{4}),$
$y(0) = \frac{3}{5}, y'(0) = -\frac{3}{5}$

8. $y'' + 3y' + 2y = 10(\sin t + \delta(t - 1)), \quad y(0) = 1,$
$y'(0) = -1$

9. $y'' + 2y' + 2y = [1 - u(t - 2)]e^t - e^2\delta(t - 2),$
$y(0) = 0, y'(0) = 1$

10. $y'' + 5y' + 6y = \delta(t - \frac{1}{2}\pi) + u(t - \pi) \cos t,$
$y(0) = 0, y'(0) = 0$

11. $y'' + 3y' + 2y = u(t - 1) + \delta(t - 2),$
$y(0) = 0, y'(0) = 1$

12. $y'' + 2y' + 5y = 25t - 100\delta(t - \pi), \quad y(0) = -2,$
$y'(0) = 5$

13. PROJECT. Heaviside Formulas. (a) Show that for a simple root a and fraction $A/(s - a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s - a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s - a)^m} + \frac{A_{m-1}}{(s - a)^{m-1}} + \cdots$$

$$+ \frac{A_1}{s - a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s - a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m - k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s - a)^m F(s)}{G(s)}\right],$$

$$k = 1, \cdots, m - 1.$$

14. TEAM PROJECT. Laplace Transform of Periodic Functions

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\mathscr{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0). \tag{11}$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n - 1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of $\sin \omega t$ in Fig. 137 has the Laplace transform

$$\mathscr{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})}$$

$$= \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 138.)

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

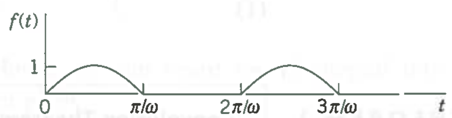

Fig. 137. Half-wave rectification

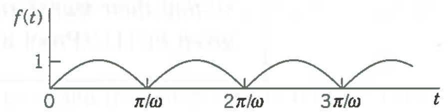

Fig. 138. Full-wave rectification

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 139.

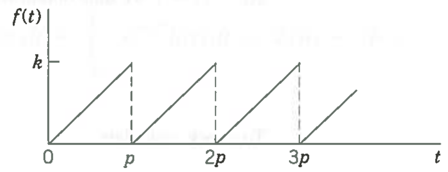

Fig. 139. Saw-tooth wave

15. Staircase function. Find the Laplace transform of the staircase function in Fig. 140 by noting that it is the difference of kt/p and the function in 14(d).

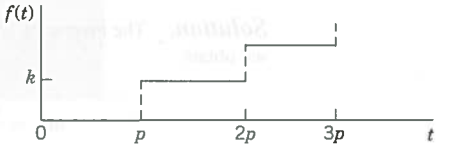

Fig. 140. Staircase function

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}).$$

Figure 143 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why?), and finally decreases to zero in a monotone fashion. ■

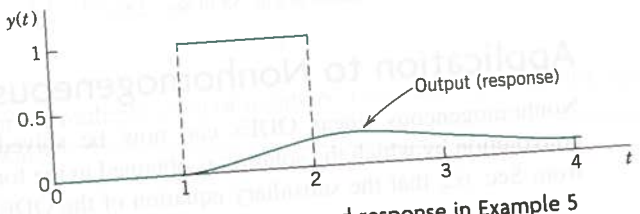

Fig. 143. Square wave and response in Example 5

Integral Equations

Convolution also helps in solving certain **integral equations**, that is, equations in which the unknown function $y(t)$ appears in an integral (and perhaps also outside of it). This concerns equations with an integral of the form of a convolution. Hence these are special and it suffices to explain the idea in terms of two examples and add a few problems in the problem set.

EXAMPLE 6 A Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau) \sin(t - \tau)\, d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing $Y = \mathscr{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad \text{and gives the answer} \qquad y(t) = t + \frac{t^3}{6}.$$

Check the result by a CAS or by substitution and repeated integration by parts (which will need patience). ■

EXAMPLE 7 Another Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation

$$y(t) - \int_0^t (1 + \tau) y(t - \tau)\, d\tau = 1 - \sinh t.$$

[3] If the upper limit of integration is *variable*, the equation is named after the Italian mathematician VITO VOLTERRA (1860–1940), and if that limit is *constant*, the equation is named after the Swedish mathematician ERIK IVAR FREDHOLM (1866–1927). "Of the second kind (first kind)" indicates that y occurs (does not occur) outside of the integral.

Solution. By (1) we can write $y - (1 + t) * y = 1 - \sinh t$. Writing $Y = \mathscr{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2 - 1}, \qquad \text{hence} \qquad Y(s) \cdot \frac{s^2 - s - 1}{s^2} = \frac{s^2 - 1 - s}{s(s^2 - 1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2 - 1} \qquad \text{and the solution is} \qquad y(t) = \cosh t.$$ ■

PROBLEM SET 6.5

1–7 CONVOLUTIONS BY INTEGRATION

Find:

1. $1 * (-1)$
2. $1 * \sin \omega t$
3. $e^{-t} * e^t$
4. $(\cos \omega t) * (\cos \omega t)$
5. $(\cos \omega t) * 1$
6. $e^{at} * e^{bt} (a \neq b)$
7. $t * e^{-t}$

8–14 INTEGRAL EQUATIONS

Solve by the Laplace transform, showing the details:

8. $y(t) + 4\int_0^t y(\tau)(t - \tau)\, d\tau = 2t$
9. $y(t) + \int_0^t y(\tau)\, d\tau = 2$
10. $y(t) - \int_0^t y(\tau) \sin 2(t - \tau)\, d\tau = \sin 2t$
11. $y(t) - \int_0^t (t - \tau) y(\tau)\, d\tau = 1$
12. $y(t) + \int_0^t y(\tau) \cosh(t - \tau)\, d\tau = t + e^t$
13. $y(t) + 2e^t \int_0^t y(\tau) e^{-\tau}\, d\tau = te^t$
14. $y(t) - \int_0^t y(\tau)(t - \tau)\, d\tau = 2 - \frac{1}{2}t^2$

15. **CAS EXPERIMENT. Variation of a Parameter.**
(a) Replace 2 in Prob. 13 by a parameter k and investigate graphically how the solution curve changes if you vary k, in particular near $k = -2$.
(b) Make similar experiments with an integral equation of your choice whose solution is oscillating.

16. **TEAM PROJECT. Properties of Convolution.** Prove:
(a) Commutativity, $f * g = g * f$
(b) Associativity, $(f * g) * v = f * (g * v)$
(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$
(d) **Dirac's delta.** Derive the sifting formula (4) in Sec. 6.4 by using f_k with $a = 0$ [(1), Sec. 6.4] and applying the mean value theorem for integrals.
(e) **Unspecified driving force.** Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \quad y(0) = K_1, \quad y'(0) = K_2$$

with $\omega \neq 0$ and an unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega} \sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega} \sin \omega t.$$

17–26 INVERSE TRANSFORMS BY CONVOLUTION

Showing details, find $f(t)$ if $\mathscr{L}(f)$ equals:

17. $\dfrac{5.5}{(s + 1.5)(s - 4)}$
18. $\dfrac{1}{(s - a)^2}$
19. $\dfrac{2\pi s}{(s^2 + \pi^2)^2}$
20. $\dfrac{9}{s(s + 3)}$
21. $\dfrac{\omega}{s^2(s^2 - \omega^2)}$
22. $\dfrac{e^{-as}}{s(s - 2)}$
23. $\dfrac{40.5}{s(s^2 - 9)}$
24. $\dfrac{240}{(s^2 + 1)(s^2 + 25)}$
25. $\dfrac{18s}{(s^2 + 36)^2}$
26. **Partial Fractions.** Solve Probs. 17, 21, and 23 by partial fraction reduction.

14–20 **INVERSE TRANSFORMS**

Using differentiation, integration, s-shifting, or convolution, and showing the details, find $f(t)$ if $\mathcal{L}(f)$ equals:

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{s}{(s^2-4)^2}$

16. $\dfrac{2s+6}{(s^2+6s+10)^2}$

17. $\ln\dfrac{s}{s-1}$

18. $\operatorname{arccot}\dfrac{s}{\pi}$

19. $\ln\dfrac{s^2+1}{(s-1)^2}$

20. $\ln\dfrac{s+a}{s+b}$

6.7 Systems of ODEs

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients (as discussed in Sec. 4.1)

$$\text{(1)}\qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned}$$

Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) in Sec. 6.2 the subsidiary system

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned}$$

By collecting the Y_1- and Y_2-terms we have

$$\text{(2)}\qquad \begin{aligned} (a_{11}-s)Y_1 + a_{12}Y_2 &= -y_1(0) - G_1(s) \\ a_{21}Y_1 + (a_{22}-s)Y_2 &= -y_2(0) - G_2(s). \end{aligned}$$

By solving this system algebraically for $Y_1(s), Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$ of the given system (1).

Note that (1) and (2) may be written in vector form (and similarly for the systems in the examples); thus, setting $\mathbf{y} = [y_1 \ \ y_2]^\mathsf{T}$, $\mathbf{A} = [a_{jk}]$, $\mathbf{g} = [g_1 \ \ g_2]^\mathsf{T}$, $\mathbf{Y} = [Y_1 \ \ Y_2]^\mathsf{T}$, $\mathbf{G} = [G_1 \ \ G_2]^\mathsf{T}$ we have

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad \text{and} \qquad (\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G}.$$

EXAMPLE 1 **Mixing Problem Involving Two Tanks**

Tank T_1 in Fig. 144 initially contains 100 gal of pure water. Tank T_2 initially contains 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1. The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

Solution. The model is obtained in the form of two equations

$$\text{Time rate of change} = \text{Inflow/min} - \text{Outflow/min}$$

for the two tanks (see Sec. 4.1). Thus,

$$y_1' = -\tfrac{8}{100}y_1 + \tfrac{2}{100}y_2 + 6. \qquad y_2' = \tfrac{8}{100}y_1 - \tfrac{8}{100}y_2.$$

The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this we see that the subsidiary system (2) is

$$\begin{aligned} (-0.08 - s)Y_1 + 0.02Y_2 &= -\frac{6}{s} \\ 0.08Y_1 + (-0.08 - s)Y_2 &= -150. \end{aligned}$$

We solve this algebraically for Y_1 and Y_2 by elimination (or by Cramer's rule in Sec. 7.7), and we write the solutions in terms of partial fractions,

$$\begin{aligned} Y_1 &= \frac{9s + 0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04} \\ Y_2 &= \frac{150s^2 + 12s + 0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}. \end{aligned}$$

By taking the inverse transform we arrive at the solution

$$\begin{aligned} y_1 &= 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t} \\ y_2 &= 100 + 125e^{-0.12t} - 75e^{-0.04t}. \end{aligned}$$

Figure 144 shows the interesting plot of these functions. Can you give physical explanations for their main features? Why do they have the limit 100? Why is y_2 not monotone, whereas y_1 is? Why is y_1 from some time on suddenly larger than y_2? Etc. ■

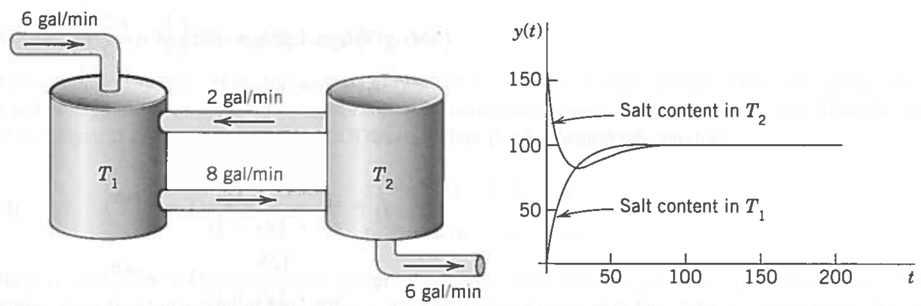

Fig. 144. Mixing problem in Example 1

Other systems of ODEs of practical importance can be solved by the Laplace transform method in a similar way, and eigenvalues and eigenvectors, as we had to determine them in Chap. 4, will come out automatically, as we have seen in Example 1.

EXAMPLE 2 **Electrical Network**

Find the currents $i_1(t)$ and $i_2(t)$ in the network in Fig. 145 with L and R measured in terms of the usual units (see Sec. 2.9), $v(t) = 100$ volts if $0 \le t \le 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.

Solution. The model of the network is obtained from Kirchhoff's Voltage Law as in Sec. 2.9. For the lower circuit we obtain

$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 = 100[1 - u(t - \tfrac{1}{2})]$$

Elimination (or Cramer's rule in Sec. 7.7) yields the solution, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s+\sqrt{3k})(s^2+2k)+k(s-\sqrt{3k})}{(s^2+2k)^2-k^2} = \frac{s}{s^2+k} + \frac{\sqrt{3k}}{s^2+3k}$$

$$Y_2 = \frac{(s^2+2k)(s-\sqrt{3k})+k(s+\sqrt{3k})}{(s^2+2k)^2-k^2} = \frac{s}{s^2+k} - \frac{\sqrt{3k}}{s^2+3k}.$$

Hence the solution of our initial value problem is (Fig. 147)

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" oscillation and a "rapid" oscillation. ■

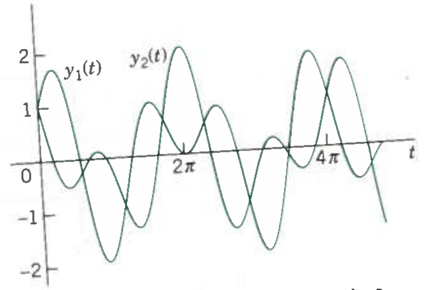

Fig. 147. Solutions in Example 3

PROBLEM SET 6.7

1. TEAM PROJECT. Comparison of Methods for Linear Systems of ODEs

(a) Models. Solve the models in Examples 1 and 2 of Sec. 4.1 by Laplace transforms and compare the amount of work with that in Sec. 4.1. Show the details of your work.

(b) Homogeneous Systems. Solve the systems (8), (11)–(13) in Sec. 4.3 by Laplace transforms. Show the details.

(c) Nonhomogeneous System. Solve the system (3) in Sec. 4.6 by Laplace transforms. Show the details.

2–15 SYSTEMS OF ODES

Using the Laplace transform and showing the details of your work, solve the IVP:

2. $y_1' - y_2 = 0,\quad y_1 + y_2' = 2\cos t,$
$y_1(0) = 1,\quad y_2(0) = 0$

3. $y_1' - 2y_1 + 3y_2 = 0,\quad y_2' - y_1 + 2y_2 = 0,$
$y_1(0) = 1,\quad y_2(0) = 0$

4. $y_1' = 4y_2 - 8\cos 4t,\quad y_2' = -3y_1 - 9\sin 4t,$
$y_1(0) = 0,\quad y_2(0) = 3$

5. $y_1' = y_2 + 2 - u(t-1),\quad y_2' = -y_1 + 1 - u(t-1),$
$y_1(0) = 1,\quad y_2(0) = 0$

6. $y_1' = 5y_1 + y_2,\quad y_2' = y_1 + 5y_2,$
$y_1(0) = 1,\quad y_2(0) = -3$

7. $y_1' = 2y_1 - 4y_2 + u(t-1)e^t,$
$y_2' = y_1 - 3y_2 + u(t-1)e^t,\quad y_1(0) = 3,\quad y_2(0) = 0$

8. $y_1' = -2y_1 + 3y_2,\quad y_2' = 4y_1 - y_2,$
$y_1(0) = 4,\quad y_2(0) = 3$

9. $y_1' = y_1 + y_2,\quad y_2' = -y_1 + 3y_2,\quad y_1(0) = 1,$
$y_2(0) = 0$

10. $y_1' = -y_2,\quad y_2' = -y_1 + 2[1 - u(t-2\pi)]\cos t,$
$y_1(0) = 1,\quad y_2(0) = 0$

11. $y_1'' = y_1 + 3y_2,\quad y_2'' = 4y_1 - 4e^t,$
$y_1(0) = 2,\quad y_1'(0) = 3,\quad y_2(0) = 1,\quad y_2'(0) = 2$

12. $y_1'' = -4y_1 + 5y_2,\quad y_2'' = -y_1 + 2y_2,$
$y_1(0) = 1,\quad y_1'(0) = 0,\quad y_2(0) = 2,\quad y_2'(0) = 0$

13. $y_1'' + y_2 = -101\sin 10t,\quad y_2'' + y_1 = 101\sin 10t,$
$y_1(0) = 0,\quad y_1'(0) = 6,\quad y_2(0) = 8,\quad y_2'(0) = -6$

14. $4y_1' + y_2' - 2y_3' = 0,\quad -2y_1' + y_3' = 1,$
$2y_2' - 4y_3' = -16t$
$y_1(0) = 2,\quad y_2(0) = 0,\quad y_3(0) = 0$

15. $-y_1' + y_2' = 2\cosh t,\quad y_2' - y_3' = e^{-t},$
$y_3' + y_1' = 2e^{-t},\quad y_1(0) = 0,\quad y_2(0) = 0,$
$y_3(0) = 1$

FURTHER APPLICATIONS

16. Forced vibrations of two masses. Solve the model in Example 3 with $k = 4$ and initial conditions $y_1(0) = 1$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2' = -1$ under the assumption that the force $11 \sin t$ is acting on the first body and the force $-11 \sin t$ on the second. Graph the two curves on common axes and explain the motion physically.

17. CAS Experiment. Effect of Initial Conditions. In Prob. 16, vary the initial conditions systematically, describe and explain the graphs physically. The great variety of curves will surprise you. Are they always periodic? Can you find empirical laws for the changes in terms of continuous changes of those conditions?

18. Mixing problem. What will happen in Example 1 if you double all flows (in particular, an increase to 12 gal/min containing 12 lb of salt from the outside), leaving the size of the tanks and the initial conditions as before? First guess, then calculate. Can you relate the new solution to the old one?

19. Electrical network. Using Laplace transforms, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 148, where $v(t) = 390 \cos t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state?

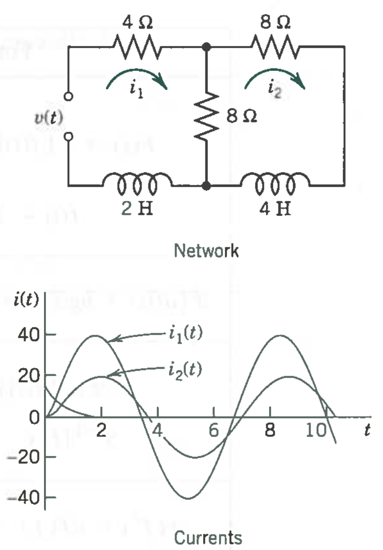

Fig. 148. Electrical network and currents in Problem 19

20. Single cosine wave. Solve Prob. 19 when the EMF (electromotive force) is acting from 0 to 2π only. Can you do this just by looking at Prob. 19, practically without calculation?

Table of Laplace Transforms (continued)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
21	$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6
22	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t}{2\omega}\sin \omega t$	6.6
23	$\frac{s^2}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	6.6
24	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2-a^2}(\cos at - \cos bt)$	
25	$\frac{1}{s^4+4k^4}$	$\frac{1}{4k^3}(\sin kt \cosh kt - \cos kt \sinh kt)$	
26	$\frac{s}{s^4+4k^4}$	$\frac{1}{2k^2}\sin kt \sinh kt$	
27	$\frac{1}{s^4-k^4}$	$\frac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\frac{s}{s^4-k^4}$	$\frac{1}{2k^2}(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\frac{a-b}{2}t\right)$	I 5.5
31	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	J 5.4
32	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	
33	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	I 5.5
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\frac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	J 5.4
37	$\frac{1}{\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\frac{1}{s^{3/2}}e^{k/s}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k>0)$	$\frac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$	

(continued)

Table of Laplace Transforms (continued)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
40	$\frac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	γ 5.5
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	6.6
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t}\sin \omega t$	
45	$\frac{1}{s}\operatorname{arccot} s$	$\mathrm{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. State the Laplace transforms of a few simple functions from memory.
2. What are the steps of solving an ODE by the Laplace transform?
3. In what cases of solving ODEs is the present method preferable to that in Chap. 2?
4. What property of the Laplace transform is crucial in solving ODEs?
5. Is $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$? $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
6. When and how do you use the unit step function and Dirac's delta?
7. If you know $f(t) = \mathcal{L}^{-1}\{F(s)\}$, how would you find $\mathcal{L}^{-1}\{F(s)/s^2\}$?
8. Explain the use of the two shifting theorems from memory.
9. Can a discontinuous function have a Laplace transform? Give reason.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

11–19 LAPLACE TRANSFORMS

Find the transform, indicating the method used and showing the details.

11. $3\cosh t - 5\sinh 2t$
12. $e^{-2t}(\cos 2t - 4\sin 2t)$
13. $\cos^2(\frac{1}{2}\pi t)$
14. $16t^2 u(t - \frac{1}{4})$
15. $e^{-t/2}u(t-2)$
16. $u(t-2\pi)\cos 2t$
17. $\cos t - t\sin t$
18. $(\sin \omega t) * (\cos \omega t)$
19. $4t * e^{-2t}$

20–28 INVERSE LAPLACE TRANSFORM

Find the inverse transform, indicating the method used and showing the details:

20. $\frac{7.5}{s^2 - 2s - 8}$
21. $\frac{s-1}{s^2}e^{-s}$
22. $\frac{\frac{1}{16}}{s^2 + s + \frac{1}{2}}$
23. $\frac{s\sin\theta + \omega\cos\theta}{s^2 + \omega^2}$
24. $\frac{s^2 - 6.25}{(s^2 + 6.25)^2}$
25. $\frac{2(1-s)}{s^3}$
26. $\frac{2s - 10}{s^3}e^{-5s}$
27. $\frac{2s+1}{s^2 + 2s + 5}$
28. $\frac{3s}{s^2 - 2s + 2}$

29–37 ODEs AND SYSTEMS

Solve by the Laplace transform, showing the details and graphing the solution:

29. $y'' + 2y' + 5y = 25t, \quad y(0) = -2, \quad y'(0) = -5$
30. $y'' + 16y = 4\delta(t - \pi), \quad y(0) = -1, \quad y'(0) = 0$

1. $y'' + y - 2y = 30u(t-\pi)\cos t, \quad y(0) = \frac{1}{2},$ $y'(0) = -1$

2. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \quad y(0) = 1,$ $y'(0) = 0$

3. $y'' - 5y' + 6y = 6u(t-1), \quad y(0) = 0, \quad y'(0) = 0$

4. $y_1' = y_2, \quad y_2' = -4y_1 + \delta(t-\pi), \quad y_1(0) = 0,$ $y_2(0) = 0$

35. $y_1' = 2y_1 - 4y_2, \quad y_2' = y_1 - 3y_2, \quad y_1(0) = 3,$ $y_2(0) = 0$

36. $y_1' = 2y_1 + 4y_2, \quad y_2' = y_1 + 2y_2, \quad y_1(0) = -4,$ $y_2(0) = -4$

37. $y_1' = y_2' + u(t-\pi), \quad y_2' = +y_1 + u(t+\pi),$ $y_1(0) = 1, \quad y_2(0) = 0$

38–45 MASS–SPRING SYSTEMS, CIRCUITS, NETWORKS

Model and solve by the Laplace transform:

38. Show that the model of the mechanical system in Fig. 149 (no friction, no damping) is

$$m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$
$$m_2 y_2'' = -k_2(y_2 - y_1) - k_3 y_2.$$

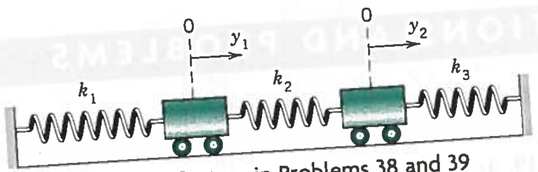

Fig. 149. System in Problems 38 and 39

39. In Prob. 38, let $m_1 = m_2 = 10$ kg, $k_1 = k_3 = 20$ kg/sec², $k_2 = 40$ kg/sec². Find the solution satisfying the initial conditions $y_1(0) = y_2(0) = 0$, $y_1'(0) = 1$ meter/sec, $y_2'(0) = -1$ meter/sec.

40. Find the model (the system of ODEs) in Prob. 38 extended by adding another mass m_3 and another spring of modulus k_4 in series.

41. Find the current $i(t)$ in the RC-circuit in Fig. 150, where $R = 10\ \Omega$, $C = 0.1$ F, $v(t) = 10t$ V if $0 < t < 4$, $v(t) = 40$ V if $t > 4$, and the initial charge on the capacitor is 0.

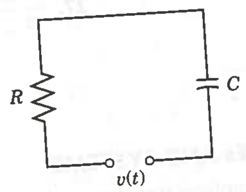

Fig. 150. RC-circuit

42. Find and graph the charge $q(t)$ and the current $i(t)$ in the LC-circuit in Fig. 151, assuming $L = 1$ H, $C = 1$ F, $v(t) = 1 - e^{-t}$ if $0 < t < \pi$, $v(t) = 0$ if $t > \pi$, and zero initial current and charge.

43. Find the current $i(t)$ in the RLC-circuit in Fig. 152, where $R = 160\ \Omega$, $L = 20$ H, $C = 0.002$ F, $v(t) = 37 \sin 10t$ V, and current and charge at $t = 0$ are zero.

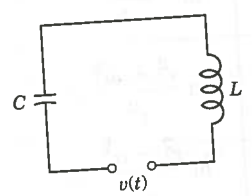

Fig. 151. LC-circuit

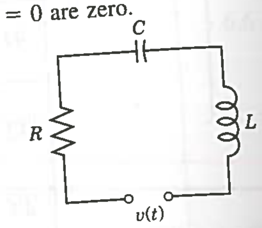

Fig. 152. RLC-circuit

44. Show that, by Kirchhoff's Voltage Law (Sec. 2.9), the currents in the network in Fig. 153 are obtained from the system

$$Li_1' + R(i_1 - i_2) = v(t)$$
$$R(i_2' - i_1') + \frac{1}{C}i_2 = 0.$$

Solve this system, assuming that $R = 10\ \Omega$, $L = 20$ H, $C = 0.05$ F, $v = 20$ V, $i_1(0) = 0$, $i_2(0) = 2$ A.

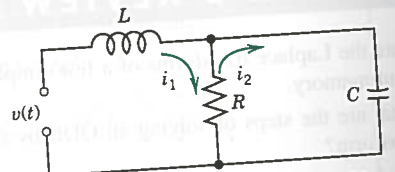

Fig. 153. Network in Problem 44

45. Set up the model of the network in Fig. 154 and find the solution, assuming that all charges and currents are 0 when the switch is closed at $t = 0$. Find the limits of $i_1(t)$ and $i_2(t)$ as $t \to \infty$, (i) from the solution, (ii) directly from the given network.

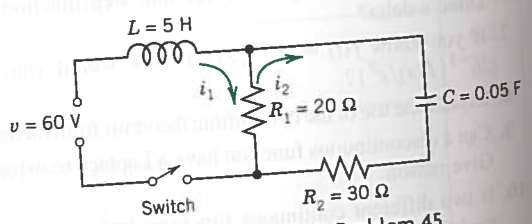

Fig. 154. Network in Problem 45

SUMMARY OF CHAPTER 6
Laplace Transforms

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{(Sec. 6.1).} \tag{1}$$

This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$\begin{aligned} \mathcal{L}(f') &= s\mathcal{L}(f) - f(0) \\ \mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0) \end{aligned} \qquad \text{(Sec. 6.2)} \tag{2}$$

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \qquad (a, b \text{ constant}) \tag{3}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subsidiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0). \tag{4}$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the small table in Sec. 6.1 or the larger table in Sec. 6.9. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the **inverse transform** $y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem. This is generally the hardest step, and in it we may again use one of those two tables. $Y(s)$ will often be a rational function, so that we can obtain the inverse $\mathcal{L}^{-1}(Y)$ by partial fraction reduction (Sec. 6.4) if we see no simpler way.

The Laplace method avoids the determination of a general solution of the homogeneous ODE, and we also need not determine values of arbitrary constants in a general solution from initial conditions; instead, we can insert the latter directly into (4). Two further facts account for the practical importance of the Laplace transform. First, it has some basic properties and resulting techniques that simplify the determination of transforms and inverses. The most important of these properties are listed in Sec. 6.8, together with references to the corresponding sections. More on the use of unit step functions and Dirac's delta can be found in Secs. 6.3 and 6.4, and more on convolution in Sec. 6.5. Second, due to these properties, the present method is particularly suitable for handling right sides $r(t)$ given by different expressions over different intervals of time, for instance, when $r(t)$ is a square wave or an impulse or of a form such as $r(t) = \cos t$ if $0 \leqq t \leqq 4\pi$ and 0 elsewhere.

The application of the Laplace transform to systems of ODEs is shown in Sec. 6.7. (The application to PDEs follows in Sec. 12.12.)