(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots .$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots . \quad ■$$

If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

PROBLEM SET 16.1

1–4 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\dfrac{\cos z}{z^4}$
2. $\dfrac{\exp(-1/z^2)}{z^2}$
3. $z^3 \cosh \dfrac{1}{z}$
4. $\dfrac{e^z}{z^2 - z^3}$

5–9 LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

5. $\dfrac{e^z}{(z-1)^2}, \quad z_0 = 1$
6. $\dfrac{1}{z^2(z-i)}, \quad z_0 = i$
7. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}, \quad z_0 = \frac{1}{4}\pi$
8. **CAS PROJECT. Partial Fractions.** Write a program for obtaining Laurent series by the use of partial fractions. Using the program, verify the calculations in Example 5 of the text. Apply the program to two other functions of your choice.
9. **TEAM PROJECT. Laurent Series. (a) Uniqueness.** Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

(b) Accumulation of singularities. Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$? (Give a reason.)

(c) Integrals. Expand the following functions in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}\,dt, \qquad \frac{1}{z^3}\int_0^z \frac{\sin t}{t}\,dt.$$

10–13 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

10. $\dfrac{1}{1 - z^2}, \quad z_0 = 0$
11. $\dfrac{1}{z}, \quad z_0 = 1$
12. $\dfrac{1}{z^2}, \quad z_0 = i$
13. $\dfrac{z^8}{1 - z^4}, \quad z_0 = 0$

16.2 Singularities and Zeros. Infinity

Roughly, a *singular point* of an analytic function $f(z)$ is a z_0 at which $f(z)$ ceases to be analytic, and a *zero* is a z at which $f(z) = 0$. Precise definitions follow below. In this section we show that Laurent series can be used for classifying singularities and Taylor series for discussing zeros.

Singularities were defined in Sec. 15.4, as we shall now recall and extend. We also remember that, by definition, a function is a ***single-valued*** relation, as was emphasized in Sec. 13.3.

We say that a function $f(z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an **isolated singularity** of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$. *Example:* $\tan z$ has isolated singularities at $\pm\pi/2, \pm 3\pi/2$, etc.; $\tan(1/z)$ has a nonisolated singularity at 0. (Explain!)

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \text{(Sec. 16.1)} \tag{1}$$

valid ***in the immediate neighborhood*** of the singular point $z = z_0$, except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

The sum of the first series is analytic at $z = z_0$, as we know from the last section. The second series, containing the negative powers, is called the **principal part** of (1), as we remember from the last section. If it has only finitely many terms, it is of the form

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0). \tag{2}$$

Then the singularity of $f(z)$ at $z = z_0$ is called a **pole**, and m is called its **order**. Poles of the first order are also known as **simple poles**.

If the principal part of (1) has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an **isolated essential singularity**.

We leave aside nonisolated singularities.

EXAMPLE 1 Poles. Essential Singularities

The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots$$

and

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,z^{2n+1}} = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - + \cdots .$$

Section 16.1 provides further examples. In that section, Example 1 shows that $z^{-5}\sin z$ has a fourth-order pole at 0. Furthermore, Example 4 shows that $1/(z^3 - z^4)$ has a third-order pole at 0 and a Laurent series with

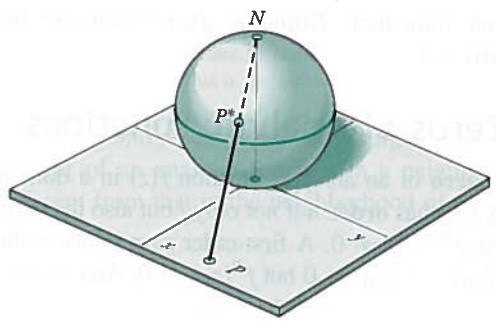

Fig. 372. Riemann sphere

Riemann Sphere. Point at Infinity

When we want to study complex functions for large $|z|$, the complex plane will generally become rather inconvenient. Then it may be better to use a representation of complex numbers on the so-called **Riemann sphere**. This is a sphere S of diameter 1 touching the complex z-plane at $z = 0$ (Fig. 372), and we let the image of a point P (a number z in the plane) be the intersection P^* of the segment PN with S, where N is the "North Pole" diametrically opposite to the origin in the plane. Then to each z there corresponds a point on S.

Conversely, each point on S represents a complex number z, except for N, which does not correspond to any point in the complex plane. This suggests that we introduce an additional point, called the **point at infinity** and denoted ∞ ("infinity") and let its image be N. The complex plane together with ∞ is called the **extended complex plane**. The complex plane is often called the *finite complex plane*, for distinction, or simply the *complex plane* as before. The sphere S is called the **Riemann sphere**. The mapping of the extended complex plane onto the sphere is known as a **stereographic projection**. (What is the image of the Northern Hemisphere? Of the Western Hemisphere? Of a straight line through the origin?)

Analytic or Singular at Infinity

If we want to investigate a function $f(z)$ for large $|z|$, we may now set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighborhood of $w = 0$. We define $f(z)$ to be **analytic or singular at infinity** if $g(w)$ is analytic or singular, respectively, at $w = 0$. We also define

$$g(0) = \lim_{w \to 0} g(w) \tag{4}$$

if this limit exists.

Furthermore, we say that $f(z)$ has an *nth-order zero at infinity* if $f(1/w)$ has such a zero at $w = 0$. Similarly for poles and essential singularities.

EXAMPLE 5 Functions Analytic or Singular at Infinity. Entire and Meromorphic Functions

The function $f(z) = 1/z^2$ is analytic at ∞ since $g(w) = f(1/w) = w^2$ is analytic at $w = 0$, and $f(z)$ has a second-order zero at ∞. The function $f(z) = z^3$ is singular at ∞ and has a third-order pole there since the function $g(w) = f(1/w) = 1/w^3$ has such a pole at $w = 0$. The function e^z has an essential singularity at ∞ since $e^{1/w}$ has such a singularity at $w = 0$. Similarly, $\cos z$ and $\sin z$ have an essential singularity at ∞.

Recall that an **entire function** is one that is analytic everywhere in the (finite) complex plane. Liouville's theorem (Sec. 14.4) tells us that the only *bounded* entire functions are the constants, hence any nonconstant entire function must be unbounded. Hence it has a singularity at ∞, a pole if it is a polynomial or an essential singularity if it is not. The functions just considered are typical in this respect.

An analytic function whose only singularities in the finite plane are poles is called a **meromorphic function**. Examples are rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$. ∎

In this section we used Laurent series for investigating singularities. In the next section we shall use these series for an elegant integration method.

PROBLEM SET 16.2

1–6 ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \frac{1}{2}z$
2. $(z + 81i)^4$
3. $\tan^2 2z$
4. $\cosh^4 z$
5. **Zeros.** If $f(z)$ is analytic and has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$ at z_0.
6. **TEAM PROJECT. Zeros. (a) Derivative.** Show that if $f(z)$ has a zero of order $n > 1$ at $z = z_0$, then $f'(z)$ has a zero of order $n - 1$ at z_0.

   (b) Poles and zeros. Prove Theorem 4.

   (c) Isolated *k*-points. Show that the points at which a nonconstant analytic function $f(z)$ has a given value k are isolated.

   (d) Identical functions. If $f_1(z)$ and $f_2(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.

7–12 SINGULARITIES

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

7. $\dfrac{1}{(z + 2i)^2} - \dfrac{z}{z - i} + \dfrac{z + 1}{(z - i)^2}$
8. $\tan \pi z$
9. $(z - \pi)^{-1} \sin z$
10. **Essential singularity.** Discuss e^{1/z^2} in a similar way as $e^{1/z}$ is discussed in Example 3 of the text.
11. **Poles.** Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.
12. **Riemann sphere.** Assuming that we let the image of the x-axis be the meridians 0° and 180°, describe and sketch (or graph) the images of the following regions on the Riemann sphere: **(a)** $|z| > 100$, **(b)** the lower half-plane, **(c)** $\frac{1}{2} \le |z| \le 2$.

16.3 Residue Integration Method

We now cover a second method of evaluating complex integrals. Recall that we solved complex integrals directly by Cauchy's integral formula in Sec. 14.3. In Chapter 15 we learned about power series and especially Taylor series. We generalized Taylor series to Laurent series (Sec. 16.1) and investigated singularities and zeroes of various functions (Sec. 16.2). Our hard work has paid off and we see how much of the theoretical groundwork comes together in evaluating complex integrals by the residue method.

The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\, dz$$

taken around a simple closed path C. The idea is as follows.

If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem (Sec. 14.2), and we are done.

The situation changes if $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before. Then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

EXAMPLE 5 **Integration by the Residue Theorem. Several Contours**

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}\,dz$$

Solution. The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

[Confirm this by (4).] *Answer:* (a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0. ∎

EXAMPLE 6 **Another Application of the Residue Theorem**

Integrate $(\tan z)/(z^2-1)$ counterclockwise around the circle C: $|z| = \frac{3}{2}$.

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C. Because of the denominator $z^2 - 1 = (z-1)(z+1)$ the given function has simple poles at ± 1. We thus obtain from (4) and the residue theorem

$$\oint_C \frac{\tan z}{z^2-1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=1} \frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1} \frac{\tan z}{z^2-1}\right)$$

$$= 2\pi i\left(\left.\frac{\tan z}{2z}\right|_{z=1} + \left.\frac{\tan z}{2z}\right|_{z=-1}\right)$$

$$= 2\pi i \tan 1 = 9.7855i. \quad ∎$$

EXAMPLE 7 **Poles and Essential Singularities**

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz.$$

Solution. Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4); note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16},$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2}\cdot\frac{1}{z} + \cdots \qquad (|z| > 0).$$

Answer: $2\pi i(-\frac{1}{16} - \frac{1}{16} + \frac{1}{2}\pi^2) = \pi(\pi^2 - \frac{1}{4})i = 30.221i$ by the residue theorem. ∎

PROBLEM SET 16.3

1–5 RESIDUES

Find all the singularities in the finite plane and the corresponding residues. Show the details.

1. $\dfrac{\sin 2z}{z^6}$
2. $\dfrac{8}{1+z^2}$
3. $\tan z$
4. $e^{1/(1-z)}$
5. **CAS PROJECT. Residue at a Pole.** Write a program for calculating the residue at a pole of any order in the finite plane. Use it for solving Probs. 5–6 (and online Probs. 7–10).

6–10 RESIDUE INTEGRATION

Evaluate (counterclockwise). Show the details.

6. $\displaystyle\oint_C \frac{z-23}{z^2-4z-5}\,dz, \quad C: |z-2-i| = 3.2$
7. $\displaystyle\oint_C \tan 2\pi z\,dz, \quad C: |z-0.2| = 0.2$
8. $\displaystyle\oint_C e^{1/z}\,dz, \quad C$: the unit circle
9. $\displaystyle\oint_C \frac{\exp(-z^2)}{\sin 4z}\,dz, \quad C: |z| = 1.5$
10. $\displaystyle\oint_C \frac{z\cosh \pi z}{z^4+13z^2+36}\,dz, \quad |z| = \pi$

16.4 Residue Integration of Real Integrals

Surprisingly, residue integration can also be used to evaluate certain classes of complicated real integrals. This shows an advantage of complex analysis over real analysis or calculus.

Integrals of Rational Functions of cos θ and sin θ

We first consider integrals of the type

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta \tag{1}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite (does not become infinite) on the interval of integration. Setting $e^{i\theta} = z$, we obtain

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right). \end{aligned} \tag{2}$$

semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

by Theorem 1. Together this shows that the principal value P of the integral from $-\infty$ to ∞ plus K equals J; hence $P = J - K = J + \pi i \operatorname{Res}_{z=a} f(z)$. If $f(z)$ has several simple poles on the real axis, then K will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

$$\text{(14)} \qquad \text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

EXAMPLE 4 Poles on the Real Axis

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i \left(\frac{3-i}{20}\right) + \pi i \left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10}. \qquad \blacksquare$$

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.

PROBLEM SET 16.4

1–4 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals and show the details of your work.

1. $\displaystyle\int_0^{\pi} \frac{2\,d\theta}{k - \cos\theta}$
2. $\displaystyle\int_0^{2\pi} \frac{1 + \sin\theta}{3 + \cos\theta}\,d\theta$
3. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$
4. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\,d\theta$

5–8 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your work.

5. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3}$
6. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1}\,dx$
7. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$
8. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{8 - x^3}\,dx$

9–14 IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

9. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$
10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$
11. $\displaystyle\int_{-\infty}^{\infty} \frac{x + 5}{x^3 - x}\,dx$
12. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\,dx$
13. **CAS EXPERIMENT. Simple Poles on the Real Axis.** Experiment with integrals $\int_{-\infty}^{\infty} f(x)\,dx$, $f(x) = [(x - a_1)(x - a_2)\cdots(x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.
14. **TEAM PROJECT. Comments on Real Integrals.** **(a) Formula (10)** follows from (9). Give the details.

    (b) Use of auxiliary results. Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a, a, a + ib, -a + ib$, letting $a \to \infty$, and using

    $$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

    show that

    $$\int_0^{\infty} e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

    (This integral is needed in heat conduction in Sec. 12.7.)

    (c) Inspection. Solve online Probs. 1 and 2 without calculation.

CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS

1. What is a Laurent series? Its principal part? Its use? Give simple examples.
2. What kind of singularities did we discuss? Give definitions and examples.
3. What is the residue? Its role in integration? Explain methods to obtain it.
4. Can the residue at a singularity be zero? At a simple pole? Give reason.
5. State the residue theorem and the idea of its proof from memory.
6. How did we evaluate real integrals by residue integration? How did we obtain the closed paths needed?
7. What are improper integrals? Their principal value? Why did they occur in this chapter?
8. What do you know about zeros of analytic functions? Give examples.
9. What is the extended complex plane? The Riemann sphere R? Sketch $z = 1 + i$ on R.
10. What is an entire function? Can it be analytic at infinity? Explain the definitions.