AMPLE 1 Application of Theorem 3

Find the radius of convergence R of the following series by applying Theorem 3.

$$\sum_{n=2}^{\infty} \binom{n}{2} z^n = z^2 + 3z^3 + 6z^4 + 10z^5 + \cdots.$$

Solution. Differentiate the geometric series twice term by term and multiply the result by $z^2/2$. This yields the given series. Hence $R = 1$ by Theorem 3. ■

EOREM 4

Termwise Integration of Power Series

The power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \cdots$ *term by term has the same radius of convergence as the original series.*

The proof is similar to that of Theorem 3.

With the help of Theorem 3, we establish the main result in this section.

Power Series Represent Analytic Functions

HEOREM 5

Analytic Functions. Their Derivatives

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, by the first statement, each of them represents an analytic function.

PROOF (a) We consider any power series (1) with positive radius of convergence R. Let $f(z)$ be its sum and $f_1(z)$ the sum of its derived series; thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{4}$$

We show that $f(z)$ is analytic and has the derivative $f_1(z)$ in the interior of the circle of convergence. We do this by proving that for any fixed z with $|z| < R$ and $\Delta z \to 0$ the difference quotient $[f(z + \Delta z) - f(z)]/\Delta z$ approaches $f_1(z)$. By termwise addition we first have from (4)

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \left[\frac{(z + \Delta z)^n - z^n}{\Delta z} - n z^{n-1}\right]. \tag{5}$$

Note that the summation starts with 2, since the constant term drops out in taking the difference $f(z + \Delta z) - f(z)$, and so does the linear term when we subtract $f_1(z)$ from the difference quotient.

(b) We claim that the series in (5) can be written

$$\sum_{n=2}^{\infty} a_n \Delta z[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n-2)z^{n-3}(z + \Delta z) + (n-1)z^{n-2}]. \tag{6}$$

The somewhat technical proof of this is given in App. 4.

(c) We consider (6). The brackets contain $n - 1$ terms, and the largest coefficient is $n - 1$. Since $(n-1)^2 \leqq n(n-1)$, we see that for $|z| \leqq R_0$ and $|z + \Delta z| \leqq R_0$, $R_0 < R$, the absolute value of this series (6) cannot exceed

$$|\Delta z| \sum_{n=2}^{\infty} |a_n| n(n-1) R_0^{n-2}. \tag{7}$$

This series with a_n instead of $|a_n|$ is the second derived series of (2) at $z = R_0$ and converges absolutely by Theorem 3 of this section and Theorem 1 of Sec. 15.2. Hence our present series (7) converges. Let the sum of (7) (without the factor $|\Delta z|$) be $K(R_0)$. Since (6) is the right side of (5), our present result is

$$\left|\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z)\right| \leqq |\Delta z| K(R_0).$$

Letting $\Delta z \to 0$ and noting that $R_0 (< R)$ is arbitrary, we conclude that $f(z)$ is analytic at any point interior to the circle of convergence and its derivative is represented by the derived series. From this the statements about the higher derivatives follow by induction. ■

Summary. The results in this section show that power series are about as nice as we could hope for: we can differentiate and integrate them term by term (Theorems 3 and 4). Theorem 5 accounts for the great importance of power series in complex analysis: the sum of such a series (with a positive radius of convergence) is an analytic function and has derivatives of all orders, which thus in turn are analytic functions. But this is only part of the story. In the next section we show that, conversely, *every* given analytic function $f(z)$ can be represented by power series, called ***Taylor series*** and being the complex analog of the real Taylor series of calculus.

PROBLEM SET 15.3

1. **Relation to Calculus.** Material in this section generalizes calculus. Give details.
2. **Termwise addition.** Write out the details of the proof on termwise addition and subtraction of power series.
3. **On Theorem 3.** Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$, as claimed.
4. **Cauchy product.** Show that $(1 - z)^{-2} = \sum_{n=0}^{\infty} (n+1)z^n$ (a) by using the Cauchy product, (b) by differentiating a suitable series.

5–15 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION

Find the radius of convergence in two ways: (a) directly by the Cauchy–Hadamard formula in Sec. 15.2, and (b) from a series of simpler terms by using Theorem 3 or Theorem 4.

5. $\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n} (z - 2i)^n$
6. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{z}{2\pi}\right)^{2n+1}$
7. $\sum_{n=1}^{\infty} \frac{n}{5^n} (z + 2i)^{2n}$
8. $\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$

$\sum_{n=1}^{\infty} \frac{(-3)^n}{n(n+1)(n+2)} z^{2n}$

$\sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{z}{2}\right)^n$

$\sum_{n=1}^{\infty} \frac{2^n n(n+1)}{5^n} z^{2n}$

$\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$

$\sum_{n=0}^{\infty} \left[\binom{n+k}{k}\right]^{-1} z^{n+k}$

$\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

$\sum_{n=2}^{\infty} \frac{5^n n(n-1)}{3^n} (z-i)^n$

20 APPLICATIONS OF THE IDENTITY THEOREM

clearly and explicitly where and how you are using rem 2.

Even functions. If $f(z)$ in (2) is *even* (i.e., $f(-z) = f(z)$), show that $a_n = 0$ for odd n. Give examples.

17. Odd function. If $f(z)$ in (2) is *odd* (i.e., $f(-z) = -f(z)$), show that $a_n = 0$ for even n. Give examples.

18. Binomial coefficients. Using $(1+z)^p(1+z)^q = (1+z)^{p+q}$, obtain the basic relation

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

19. Find applications of Theorem 2 in differential equations and elsewhere.

20. TEAM PROJECT. Fibonacci numbers.[2] **(a)** The Fibonacci numbers are recursively defined by $a_0 = a_1 = 1$, $a_{n+1} = a_n + a_{n-1}$ if $n = 1, 2, \cdots$. Find the limit of the sequence (a_{n+1}/a_n).

(b) Fibonacci's rabbit problem. Compute a list of $a_1, \cdots, a_{12}$. Show that $a_{12} = 233$ is the number of pairs of rabbits after 12 months if initially there is 1 pair and each pair generates 1 pair per month, beginning in the second month of existence (no deaths occurring).

(c) Generating function. Show that the *generating function* of the **Fibonacci numbers** is $f(z) = 1/(1 - z - z^2)$; that is, if a power series (1) represents this $f(z)$, its coefficients must be the Fibonacci numbers and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

.4 Taylor and Maclaurin Series

The **Taylor series**[3] of a function $f(z)$, the complex analog of the real Taylor series is

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0) \tag{1}$$

or, by (1), Sec. 14.4,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*. \tag{2}$$

In (2) we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that $f(z)$ is analytic in a domain containing C and every point inside C.

A **Maclaurin series**[3] is a Taylor series with center $z_0 = 0$.

[2] LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180–1250, Italian mathematician, credited with the first renaissance of mathematics on Christian soil.

[3] BROOK TAYLOR (1685–1731), English mathematician who introduced real Taylor series. COLIN MACLAURIN (1698–1746), Scots mathematician, professor at Edinburgh.

The **remainder** of the Taylor series (1) after the term $a_n(z - z_0)^n$ is

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)}\, dz^* \tag{3}$$

(proof below). Writing out the corresponding partial sum of (1), we thus have

$$f(z) = f(z_0) + \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z). \tag{4}$$

This is called **Taylor's formula** *with remainder.*

We see that ***Taylor series are power series.*** From the last section we know that power series represent analytic functions. And we now show that *every* analytic function can be represented by power series, namely, by Taylor series (with various centers). This makes Taylor series very important in complex analysis. Indeed, they are more fundamental in complex analysis than their real counterparts are in calculus.

THEOREM 1 **Taylor's Theorem**

Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D. Then there exists precisely one Taylor series (1) with center z_0 that represents $f(z)$. This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic. The remainders $R_n(z)$ of (1) can be represented in the form (3). The coefficients satisfy the inequality

$$|a_n| \leq \frac{M}{r^n} \tag{5}$$

where M is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.

PROOF The key tool is Cauchy's integral formula in Sec. 14.3; writing z and z^* instead of z_0 and z (so that z^* is the variable of integration), we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z}\, dz^*. \tag{6}$$

z lies inside C, for which we take a circle of radius r with center z_0 and interior in D (Fig. 367). We develop $1/(z^* - z)$ in (6) in powers of $z - z_0$. By a ***standard algebraic manipulation*** (worth remembering!) we first have

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \dfrac{z - z_0}{z^* - z_0}\right)}. \tag{7}$$

EXAMPLE 6 **Integration**

Find the Maclaurin series of $f(z) = \arctan z$.

Solution. We have $f'(z) = 1/(1 + z^2)$. Integrating (19) term by term and using $f(0) = 0$ we get

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad (|z| < 1);$$

this series represents the principal value of $w = u + iv = \arctan z$ defined as that value for which $|u| < \pi/2$. ■

EXAMPLE 7 **Development by Using the Geometric Series**

Develop $1/(c - z)$ in powers of $z - z_0$, where $c - z_0 \neq 0$.

Solution. This was done in the proof of Theorem 1, where $c = z^*$. The beginning was simple algebra and then the use of (11) with z replaced by $(z - z_0)/(c - z_0)$:

$$\frac{1}{c - z} = \frac{1}{c - z_0 - (z - z_0)} = \frac{1}{(c - z_0)\left(1 - \dfrac{z - z_0}{c - z_0}\right)} = \frac{1}{c - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n$$

$$= \frac{1}{c - z_0}\left(1 + \frac{z - z_0}{c - z_0} + \left(\frac{z - z_0}{c - z_0}\right)^2 + \cdots\right).$$

This series converges for

$$\left|\frac{z - z_0}{c - z_0}\right| < 1, \qquad \text{that is,} \qquad |z - z_0| < |c - z_0|.$$ ■

EXAMPLE 8 **Binomial Series, Reduction by Partial Fractions**

Find the Taylor series of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Solution. We develop $f(z)$ in partial fractions and the first fraction in a **binomial series**

$$(20) \qquad \frac{1}{(1+z)^m} = (1 + z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$

$$= 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

with $m = 2$ and the second fraction in a geometric series, and then add the two series term by term. This gives

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{[3 + (z-1)]^2} - \frac{2}{2 - (z-1)} = \frac{1}{9}\left(\frac{1}{[1 + \frac{1}{3}(z-1)]^2}\right) - \frac{1}{1 - \frac{1}{2}(z-1)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty} \binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 - \frac{275}{1944}(z-1)^3 - \cdots.$$

We see that the first series converges for $|z - 1| < 3$ and the second for $|z - 1| < 2$. This had to be expected because $1/(z + 2)^2$ is singular at -2 and $2/(z - 3)$ at 3, and these points have distance 3 and 2, respectively, from the center $z_0 = 1$. Hence the whole series converges for $|z - 1| < 2$. ■

PROBLEM SET 15.4

1. **Calculus.** Which of the series in this section have you discussed in calculus? What is new?

2. **On Examples 5 and 6.** Give all the details in the derivation of the series in those examples.

3–10 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence.

3. $\sin \dfrac{z^2}{2}$
4. $\dfrac{z+2}{1-z^2}$
5. $\dfrac{1}{8 + z^4}$
6. $\dfrac{1}{1 + 2iz}$
7. $2\sin^2(z/2)$
8. $\sin^2 z$
9. $\displaystyle\int_0^z \exp(-t^2)\,dt$
10. $\exp(z^2)\displaystyle\int_0^z \exp(-t^2)\,dt$

11–14 HIGHER TRANSCENDENTAL FUNCTIONS

Find the Maclaurin series by termwise integrating the integrand. (The integrals cannot be evaluated by the usual methods of calculus. They define the **error function** erf z, **sine integral** Si(z), and **Fresnel integrals**[4] S(z) and C(z), which occur in statistics, heat conduction, optics, and other applications. These are special so-called higher transcendental functions.)

11. $S(z) = \displaystyle\int_0^z \sin t^2\,dt$
12. $C(z) = \displaystyle\int_0^z \cos t^2\,dt$
13. $\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-t^2}\,dt$
14. $\operatorname{Si}(z) = \displaystyle\int_0^z \frac{\sin t}{t}\,dt$

15. **CAS Project. sec, tan. (a) Euler numbers.** The Maclaurin series

$$(21) \qquad \sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - + \cdots$$

defines the *Euler numbers* E_{2n}. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Write a program that computes the E_{2n} from the coefficient formula in (1) or extracts them as a list from the series. (For tables see Ref. [GenRef1], p. 810, listed in App. 1.)

(b) Bernoulli numbers. The Maclaurin series

$$(22) \qquad \frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots$$

defines the *Bernoulli numbers* B_n. Using undetermined coefficients, show that

$$(23) \qquad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0,$$
$$B_4 = -\tfrac{1}{30}, \quad B_5 = 0, \quad B_6 = \tfrac{1}{42}, \cdots.$$

Write a program for computing B_n.

(c) Tangent. Using (1), (2), Sec. 13.6, and (22), show that $\tan z$ has the following Maclaurin series and calculate from it a table of $B_0, \cdots, B_{20}$:

$$(24) \qquad \tan z = \frac{2i}{e^{2iz} - 1} - \frac{4i}{e^{4iz} - 1} - i$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_{2n} z^{2n-1}.$$

16. **Inverse sine.** Developing $1/\sqrt{1 - z^2}$ and integrating, show that

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots \quad (|z| < 1).$$

Show that this series represents the principal value of $\arcsin z$ (defined in Team Project 30, Sec. 13.7).

17. **TEAM PROJECT. Properties from Maclaurin Series.** Clearly, from series we can compute function values. In this project we show that properties of functions can often be discovered from their Taylor or Maclaurin series. Using suitable series, prove the following.

(a) The formulas for the derivatives of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, and $\operatorname{Ln}(1 + z)$

(b) $\frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

(c) $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$

18–25 TAYLOR SERIES

Find the Taylor series with center z_0 and its radius of convergence.

18. $1/z$, $z_0 = i$
19. $1/(1 + z)$, $z_0 = -i$
20. $\cos^2 z$, $z_0 = \pi/2$
21. $\cos z$, $z_0 = \pi$
22. $\cosh(z - \pi i)$, $z_0 = \pi i$
23. $1/(z - i)^2$, $z_0 = -i$
24. $e^{z(z-2)}$, $z_0 = 1$
25. $\sinh(2z - i)$, $z_0 = i/2$

[4] AUGUSTIN FRESNEL (1788–1827), French physicist and engineer, known for his work in optics.

EXAMPLE 4 **Weierstrass *M*-Test**

Does the following series converge uniformly in the disk $|z| \leqq 1$?

$$\sum_{m=1}^{\infty} \frac{z^m + 1}{m^2 + \cosh m|z|}.$$

Solution. Uniform convergence follows by the Weierstrass M-test and the convergence of $\Sigma 1/m^2$ (see Sec. 15.1, in the proof of Theorem 8) because

$$\left|\frac{z^m + 1}{m^2 + \cosh m|z|}\right| \leqq \frac{|z|^m + 1}{m^2} \leqq \frac{2}{m^2}. \qquad \blacksquare$$

No Relation Between Absolute and Uniform Convergence

We finally show the surprising fact that there are series that converge absolutely but not uniformly, and others that converge uniformly but not absolutely, so that there is no relation between the two concepts.

EXAMPLE 5 **No Relation Between Absolute and Uniform Convergence**

The series in Example 2 converges absolutely but not uniformly, as we have shown. On the other hand, the series

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{x^2 + m} = \frac{1}{x^2 + 1} - \frac{1}{x^2 + 2} + \frac{1}{x^2 + 3} - + \cdots \qquad (x \text{ real})$$

converges uniformly on the whole real line but not absolutely.

Proof. By the familiar Leibniz test of calculus (see App. A3.3) the remainder R_n does not exceed its first term in absolute value, since we have a series of alternating terms whose absolute values form a monotone decreasing sequence with limit zero. Hence given $\epsilon > 0$, for all x we have

$$|R_n(x)| \leqq \frac{1}{x^2 + n + 1} < \frac{1}{n} < \epsilon \qquad \text{if } n > N(\epsilon) \geqq \frac{1}{\epsilon}.$$

This proves uniform convergence, since $N(\epsilon)$ does not depend on x.

The convergence is not absolute because for any *fixed* x we have

$$\left|\frac{(-1)^{m-1}}{x^2 + m}\right| = \frac{1}{x^2 + m} > \frac{k}{m}$$

where k is a suitable constant, and $k\Sigma 1/m$ diverges. $\blacksquare$

PROBLEM SET 15.5

1. CAS EXPERIMENT. Graphs of Partial Sums. (a) **Fig. 368.** Produce this exciting figure using your CAS. Add further curves, say, those of s_{256}, s_{1024}, etc. on the same screen.

(b) Power series. Study the nonuniformity of convergence experimentally by graphing partial sums near the endpoints of the convergence interval for real $z = x$.

2–9 POWER SERIES

Where does the power series converge uniformly? Give reason.

2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n+2}{7n-3}\right)^n z^n$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}(z + i)^{2n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n(1-i)^n}{n!}(z - i)^n$

5. $\displaystyle\sum_{n=2}^{\infty} \binom{n}{2}(4z + 2i)^n$

6. $\displaystyle\sum_{n=0}^{\infty} 2^n(\tanh n^2) z^{2n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2}\left(z + \frac{1}{4}i\right)$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)}(z - 1)^{2n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n^2}(z - 2i)^n$

10–17 UNIFORM CONVERGENCE

Prove that the series converges uniformly in the indicated region.

10. $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad |z| \leqq 10^{20}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| \leqq 1$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^3 \sinh n|z|}, \quad |z| \leqq 1$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^n |z|}{n^2}, \quad$ all z

14. $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{|z|^{2n} + 1}, \quad 2 \leqq |z| \leqq 10$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n!)} z^n, \quad |z| \leqq 3$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\tanh^n |z|}{n(n+1)}, \quad$ all z

17. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^{2n}}{n^2} z^{2n}, \quad |z| \leqq 0.25$

18. TEAM PROJECT. Uniform Convergence.

(a) Weierstrass *M*-test. Give a proof.

(b) Termwise differentiation. Derive Theorem 4 from Theorem 3.

(c) Subregions. Prove that uniform convergence of a series in a region G implies uniform convergence in any portion of G. Is the converse true?

(d) Example 2. Find the precise region of convergence of the series in Example 2 with x replaced by a complex variable z.

(e) Figure 369. Show that $x^2 \sum_{m=1}^{\infty} (1 + x^2)^{-m} = 1$ if $x \neq 0$ and 0 if $x = 0$. Verify by computation that the partial sums s_1, s_2, s_3 look as shown in Fig. 369.

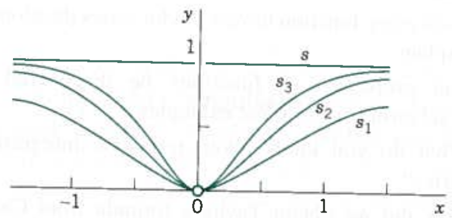

Fig. 369. Sum s and partial sums in Team Project 18(e)

19–20 HEAT EQUATION

Show that (9) in Sec. 12.6 with coefficients (10) is a solution of the heat equation for $t > 0$, assuming that $f(x)$ is continuous on the interval $0 \leqq x \leqq L$ and has one-sided derivatives at all interior points of that interval. Proceed as follows.

19. Show that $|B_n|$ is bounded, say $|B_n| < K$ for all n. Conclude that

$$|u_n| < Ke^{-\lambda_n^2 t_0} \qquad \text{if} \qquad t \geqq t_0 > 0$$

and, by the Weierstrass test, the series (9) converges uniformly with respect to x and t for $t \geqq t_0$, $0 \leqq x \leqq L$. Using Theorem 2, show that $u(x, t)$ is continuous for $t \geqq t_0$ and thus satisfies the boundary conditions (2) for $t \geqq t_0$.

20. Show that $|\partial u_n/\partial t| < \lambda_n^2 K e^{-\lambda_n^2 t_0}$ if $t \geqq t_0$ and the series of the expressions on the right converges, by the ratio test. Conclude from this, the Weierstrass test, and Theorem 4 that the series (9) can be differentiated term by term with respect to t and the resulting series has the sum $\partial u/\partial t$. Show that (9) can be differentiated twice with respect to x and the resulting series has the sum $\partial^2 u/\partial x^2$. Conclude from this and the result to Prob. 19 that (9) is a solution of the heat equation for all $t \geqq t_0$. (The proof that (9) satisfies the given initial condition can be found in Ref. [C10] listed in App. 1.)

CHAPTER 15 REVIEW QUESTIONS AND PROBLEMS

1. What is convergence test for series? State two tests from memory. Give examples.
2. What is a power series? Why are these series very important in complex analysis?
3. What is absolute convergence? Conditional convergence? Uniform convergence?
4. What do you know about convergence of power series?
5. What is a Taylor series? Give some basic examples.
6. What do you know about adding and multiplying power series?
7. Does every function have a Taylor series development? Explain.
8. Can properties of functions be discovered from Maclaurin series? Give examples.
9. What do you know about termwise integration of series?
10. How did we obtain Taylor's formula from Cauchy's formula?

11–15 RADIUS OF CONVERGENCE

Find the radius of convergence.

11. $\sum_{n=0}^{\infty} (z+1)^n$

12. $\sum_{n=2}^{\infty} \frac{4^n}{n-1}(z-\pi i)^n$

13. $\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-i)^n$

14. $\sum_{n=1}^{\infty} \frac{n^5}{n!}(z-3i)^{2n}$

15. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{3n} z^n$

16–20 RADIUS OF CONVERGENCE

Find the radius of convergence. Try to identify the sum of the series as a familiar function.

16. $\sum_{n=1}^{\infty} \frac{z^n}{n}$

17. $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} z^n$

18. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\pi z)^{2n+1}$

19. $\sum_{n=0}^{\infty} \frac{z^{n+1/2}}{(2n+1)!}$

20. $\sum_{n=0}^{\infty} \frac{z^n}{(3+4i)^n}$

21–25 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence. Show details.

21. $\cosh z^2$
22. $1/(1-z)^3$
23. $\cos(z^2)$
24. $1/(\pi z + 1)$
25. $(e^{z^2} - 1)/z^2$

26–30 TAYLOR SERIES

Find the Taylor series with the given point as center and its radius of convergence.

26. z^5, i
27. $\sin z$, π
28. $1/z$, $2i$
29. $\mathrm{Ln}\, z$, 3
30. e^z, πi

SUMMARY OF CHAPTER 15
Power Series, Taylor Series

Sequences, series, and convergence tests are discussed in Sec. 15.1. A **power series** is of the form (Sec. 15.2)

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots; \tag{1}$$

z_0 is its *center*. The series (1) converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$, where R is the **radius of convergence**. Some power series converge

for all z (then we write $R = \infty$). In exceptional cases a power series may converge only at the center; such a series is practically useless. Also, $R = \lim |a_n/a_{n+1}|$ if this limit exists. The series (1) converges absolutely (Sec. 15.2) and ***uniformly*** (Sec. 15.5) in every closed disk $|z - z_0| \leqq r < R$ $(R > 0)$. It represents an analytic function $f(z)$ for $|z - z_0| < R$. The derivatives $f'(z), f''(z), \cdots$ are obtained by termwise differentiation of (1), and these series have the same radius of convergence R as (1). See Sec. 15.3.

Conversely, *every* analytic function $f(z)$ can be represented by power series. These **Taylor series** of $f(z)$ are of the form (Sec. 15.4)

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \qquad (|z - z_0| < R), \tag{2}$$

as in calculus. They converge for all z in the open disk with center z_0 and radius generally equal to the distance from z_0 to the nearest **singularity** of $f(z)$ (point at which $f(z)$ ceases to be analytic as defined in Sec. 15.4). If $f(z)$ is **entire** (analytic for all z; see Sec. 13.5), then (2) converges for all z. The functions e^z, $\cos z$, $\sin z$, etc. have Maclaurin series, that is, Taylor series with center 0, similar to those in calculus (Sec. 15.4).