Solution. $g(z)$ is not analytic at -1 and 1. These are the points we have to watch for. We consider each circle separately.

(a) The circle $|z - 1| = 1$ encloses the point $z_0 = 1$ where $g(z)$ is not analytic. Hence in (1) we have to write

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\,\frac{1}{z-1};$$

thus

$$f(z) = \frac{z^2+1}{z+1}$$

and (1) gives

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(1) = 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

(b) gives the same as (a) by the principle of deformation of path.

(c) The function $g(z)$ is as before, but $f(z)$ changes because we must take $z_0 = -1$ (instead of 1). This gives a factor $z - z_0 = z + 1$ in (1). Hence we must write

$$g(z) = \frac{z^2+1}{z-1}\,\frac{1}{z+1};$$

thus

$$f(z) = \frac{z^2+1}{z-1}.$$

Compare this for a minute with the previous expression and then go on:

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(-1) = 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i.$$

(d) gives 0. Why? ■

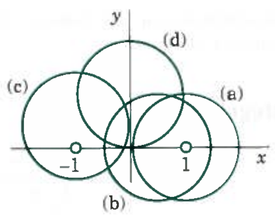


Fig. 358. Example 3

Multiply connected domains can be handled as in Sec. 14.2. For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 (Fig. 359) and z_0 is any point in that domain, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z - z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z - z_0}\,dz, \tag{3}$$

where the outer integral (over C_1) is taken counterclockwise and the inner clockwise, as indicated in Fig. 359.

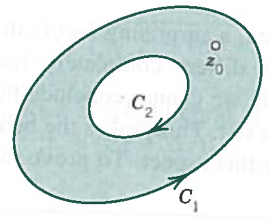


Fig. 359. Formula (3)

PROBLEM SET 14.3

1–4 CONTOUR INTEGRATION

Integrate $z^2/(z^2 - 1)$ by Cauchy's formula counterclockwise around the circle.

1. $|z + 1| = 3/2$

2. $|z - 1 - i| = \pi/2$

3. $|z + i| = 1.41$

4. $|z + 5 - 5i| = 7$

5–8 Integrate the given function around the unit circle.

5. $(\cos 2z)/4z$

6. $e^{2z}/(\pi z - i)$

7. $z^2/(4z - i)$

8. $(z \sin z)/(2z - 1)$

9. CAS EXPERIMENT. Experiment to find out to what extent your CAS can do contour integration. For this, use **(a)** the second method in Sec. 14.1 and **(b)** Cauchy's integral formula.

10. TEAM PROJECT. Cauchy's Integral Theorem. Gain additional insight into the proof of Cauchy's integral theorem by producing (2) with a contour enclosing z_0 (as in Fig. 356) and taking the limit as in the text. Choose

(a) $\displaystyle\oint_C \frac{z^3 - 6}{z - \frac{1}{2}i}\,dz$, (b) $\displaystyle\oint_C \frac{\sin z}{z - \frac{1}{2}\pi}\,dz$,

and (c) another example of your choice.

11–19 FURTHER CONTOUR INTEGRALS

Integrate counterclockwise or as indicated. Show the details.

11. $\displaystyle\oint_C \frac{dz}{z^2+4}$, $C: 4x^2 + (y-2)^2 = 4$

12. $\displaystyle\oint_C \frac{z}{z^2+4z+3}\,dz$, C the circle with center -1 and radius 2

13. $\displaystyle\oint_C \frac{z+2}{z-2}\,dz$, $C: |z - 1| = 2$

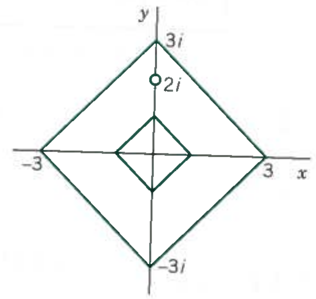
14. $\displaystyle\oint_C \frac{e^z}{ze^z - 2iz}\,dz$, $C: |z| = 0.6$

15. $\displaystyle\oint_C \frac{\cosh(z^2 - \pi i)}{z - \pi i}\,dz$, C the boundary of the square with vertices $\pm 2, \pm 2, \pm 4i$.

16. $\displaystyle\oint_C \frac{\tan z}{z - i}\,dz$, C the boundary of the triangle with vertices 0 and $\pm 1 + 2i$.

17. $\displaystyle\oint_C \frac{\operatorname{Ln}(z+1)}{z^2+1}\,dz$, $C: |z - i| = 1.4$

18. $\displaystyle\oint_C \frac{\sin z}{4z^2 - 8iz}\,dz$, C consists of the boundaries of the squares with vertices $\pm 3, \pm 3i$ counterclockwise and $\pm 1, \pm i$ clockwise (see figure).



Problem 18

19. $\displaystyle\oint_C \frac{\exp z^2}{z^2(z - 1 - i)}\,dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

20. Show that $\displaystyle\oint_C (z - z_1)^{-1}(z - z_2)^{-1}\,dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.

continuation and completion of this proof, because it implies that $(1'')$ can be proved by a similar argument, with f replaced by f', and that the general formula (1) follows by induction. ■

Applications of Theorem 1

EXAMPLE 1 Evaluation of Line Integrals

From $(1')$, for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi. \qquad ■$$

EXAMPLE 2 From $(1'')$, for any contour enclosing the point $-i$ we obtain by counterclockwise integration

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \pi i(z^4 - 3z^2 + 6)''\Big|_{z=-i} = \pi i[12z^2 - 6]_{z=-i} = -18\pi i. \qquad ■$$

EXAMPLE 3 By $(1')$, for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Bigg|_{z=1}$$

$$= 2\pi i \frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2}\Bigg|_{z=1} = \frac{6e\pi}{25} i \approx 2.050i. \qquad ■$$

Cauchy's Inequality. Liouville's and Morera's Theorems

We develop other general results about analytic functions, further showing the versatility of Cauchy's integral theorem.

Cauchy's Inequality. Theorem 1 yields a basic inequality that has many applications. To get it, all we have to do is to choose for C in (1) a circle of radius r and center z_0 and apply the ML-inequality (Sec. 14.1); with $|f(z)| \leqq M$ on C we obtain from (1)

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leqq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r.$$

This gives **Cauchy's inequality**

$$|f^{(n)}(z_0)| \leqq \frac{n! M}{r^n}. \tag{2}$$

To gain a first impression of the importance of this inequality, let us prove a famous theorem on entire functions (definition in Sec. 13.5). (For Liouville, see Sec. 11.5.)

THEOREM 2 Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

PROOF By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using (2), we see that $|f'(z_0)| < K/r$. Since $f(z)$ is entire, this holds for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = u_x + iv_x = 0$ for all z (see (4) in Sec. 13.4), hence $u_x = v_x = 0$, and $u_y = v_y = 0$ by the Cauchy–Riemann equations. Thus $u = \text{const}$, $v = \text{const}$, and $f = u + iv = \text{const}$ for all z. This completes the proof. ■

Another very interesting consequence of Theorem 1 is

THEOREM 3 Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,dz = 0 \tag{3}$$

for every closed path in D, then $f(z)$ is analytic in D.

PROOF In Sec. 14.2 we showed that if $f(z)$ is analytic in a simply connected domain D, then

$$F(z) = \int_{z_0}^{z} f(z^*)\,dz^*$$

is analytic in D and $F'(z) = f(z)$. In the proof we used only the continuity of $f(z)$ and the property that its integral around every closed path in D is zero; from these assumptions we concluded that $F(z)$ is analytic. By Theorem 1, the derivative of $F(z)$ is analytic, that is, $f(z)$ is analytic in D, and Morera's theorem is proved. ■

This completes Chapter 14.

PROBLEM SET 14.4

1–7 CONTOUR INTEGRATION. UNIT CIRCLE

Integrate counterclockwise around the unit circle.

1. $\oint_C \frac{\sin 2z}{z^4}\,dz$
2. $\oint_C \frac{z^6}{(2z-1)^6}\,dz$
3. $\oint_C \frac{e^{-z}}{z^n}\,dz, \quad n = 1, 2, \cdots$
4. $\oint_C \frac{e^z \cos z}{(z-\pi/4)^3}\,dz$
5. $\oint_C \frac{\sinh 2z}{(z-\frac{1}{2})^4}\,dz$
6. $\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2}$
7. $\oint_C \frac{\cos z}{z^{2n+1}}\,dz, \quad n = 0, 1, \cdots$

8–19 INTEGRATION. DIFFERENT CONTOURS

Integrate. Show the details. *Hint.* Begin by sketching the contour. Why?

8. $\oint_C \frac{z^3 + \sin z}{(z-i)^3}\,dz$, C the boundary of the square with vertices ± 2, $\pm 2i$ counterclockwise.
9. $\oint_C \frac{\tan \pi z}{z^2}\,dz$, C the ellipse $16x^2 + y^2 = 1$ clockwise.
10. $\oint_C \frac{4z^3 - 6}{z(z-1-i)^2}\,dz$, C consists of $|z| = 3$ counterclockwise and $|z| = 1$ clockwise.

[2]GIACINTO MORERA (1856–1909), Italian mathematician who worked in Genoa and Turin.

$\frac{1 + z) \cos z}{(2z-1)^2} dz$, $C: |z - i| = 2$ counterclockwise.

$\frac{\exp(z^2)}{z(z-2i)^2} dz$, $C: z - 3i| = 2$ clockwise.

$\frac{\text{Ln } z}{(z-4)^2} dz$, $C: |z - 3| = 2$ counterclockwise.

$\frac{\text{Ln }(z+3)}{(z-2)(z+1)^2} dz$, C the boundary of the square
h vertices ± 1.5, $\pm 1.5i$, counterclockwise.

$\frac{\cosh 4z}{(z-4)^3} dz$, C consists of $|z| = 6$ counterclock-
.e and $|z - 3| = 2$ clockwise.

$\frac{e^{4z}}{z(z-2i)^2} dz$, C consists of $|z - i| = 3$ counter-
ckwise and $|z| = 1$ clockwise.

$\frac{e^{-z} \sin z}{(z-4)^3} dz$, C consists of $|z| = 5$ counterclock-
se and $|z - 3| = \frac{3}{2}$ clockwise.

18. $\oint_C \frac{\sinh z}{z^n} dz$, $C: |z| = 1$ counterclockwise, n integer.

19. $\oint_C \frac{e^{3z}}{(4z - \pi i)^3} dz$, $C: |z| = 1$, counterclockwise.

20. TEAM PROJECT. Theory on Growth

(a) Growth of entire functions. If $f(z)$ is not a constant and is analytic for all (finite) z, and R and M are any positive real numbers (no matter how large), show that there exist values of z for which $|z| > R$ and $|f(z)| > M$. *Hint.* Use Liouville's theorem.

(b) Growth of polynomials. If $f(z)$ is a polynomial of degree $n > 0$ and M is an arbitrary positive real number (no matter how large), show that there exists a positive real number R such that $|f(z)| > M$ for all $|z| > R$.

(c) Exponential function. Show that $f(z) = e^x$ has the property characterized in (a) but does not have that characterized in (b).

(d) Fundamental theorem of algebra. *If $f(z)$ is a polynomial in z, not a constant, then $f(z) = 0$ for at least one value of z.* Prove this. *Hint.* Use (a).

APTER 14 REVIEW QUESTIONS AND PROBLEMS

hat is a parametric representation of a curve? What its advantage?

hat did we assume about paths of integration $z = z(t)$? hat is $\dot{z} = dz/dt$ geometrically?

ate the definition of a complex line integral from emory.

an you remember the relationship between complex id real line integrals discussed in this chapter?

ow can you evaluate a line integral of an analytic nction? Of an arbitrary continous complex function?

'hat value do you get by counterclockwise integration $1/z$ around the unit circle? You should remember iis. It is basic.

'hich theorem in this chapter do you regard as most nportant? State it precisely from memory.

'hat is independence of path? Its importance? State a asic theorem on independence of path in complex.

'hat is deformation of path? Give a typical example.

)on't confuse Cauchy's integral theorem (also known s *Cauchy–Goursat theorem*) and Cauchy's integral ormula. State both. How are they related?

Vhat is a doubly connected domain? How can you xtend Cauchy's integral theorem to it?

12. What do you know about derivatives of analytic functions?

13. How did we use integral formulas for derivatives in evaluating integrals?

14. How does the situation for analytic functions differ with respect to derivatives from that in calculus?

15. What is Liouville's theorem? To what complex functions does it apply?

16. What is Morera's theorem?

17. If the integrals of a function $f(z)$ over each of the two boundary circles of an annulus D taken in the same sense have different values, can $f(z)$ be analytic everywhere in D? Give reason.

18. Is $\text{Im} \oint_C f(z)\, dz = \oint_C \text{Im} f(z)\, dz$? Give reason.

19. Is $\left|\oint_C f(z)\, dz\right| = \oint_C |f(z)|\, dz$?

20. How would you find a bound for the left side in Prob. 19?

21–30 INTEGRATION

Integrate by a suitable method.

21. $\int_C z \cosh(z^2)\, dz$ from 0 to $\pi i/2$.

22. $\int_C (|z| + z)\, dz$ clockwise around the unit circle.

23. $\oint_C z^{-4} e^{-z}\, dz$ counterclockwise around $|z| = \pi$.

24. $\int_C \text{Re}\, dz$ from 0 to $2 + 8i$ along $y = 2x^2$.

25. $\int_C \frac{\tan \pi z}{(z-1)^2} dz$ clockwise around $|z - 1| = 0.1$.

26. $\int_C (z^2 + \bar{z}^2)\, dz$ from $z = 0$ horizontally to $z = 2$, then vertically upward to $2 + 2i$.

27. $\int_C (z^2 + \bar{z}^2)\, dz$ from 0 to $2 + 2i$, shortest path.

28. $\oint_C \frac{\text{Ln } z}{(z-2i)^2} dz$ counterclockwise around $|z - 1| = \frac{1}{2}$.

29. $\int_C \left(\frac{4}{z+i} + \frac{1}{z+3i}\right) dz$ clockwise around $|z - 1| = 2.5$.

30. $\int_C \cos z\, dz$ from 0 to $\frac{\pi}{2} - i$.

SUMMARY OF CHAPTER 14
Complex Integration

The **complex line integral** of a function $f(z)$ taken over a path C is denoted by

(1) $\int_C f(z)\, dz$ or, if C is closed, also by $\oint_C f(z)$ (Sec. 14.1).

If $f(z)$ is analytic in a simply connected domain D, then we can evaluate (1) as in calculus by indefinite integration and substitution of limits, that is,

(2) $$\int_C f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

for every path C in D from a point z_0 to a point z_1 (see Sec. 14.1). These assumptions imply **independence of path**, that is, (2) depends only on z_0 and z_1 (and on $f(z)$, of course) but not on the choice of C (Sec. 14.2). The existence of an $F(z)$ such that $F'(z) = f(z)$ is proved in Sec. 14.2 by Cauchy's integral theorem (see below).

A general method of integration, not restricted to analytic functions, uses the equation $z = z(t)$ of C, where $a \leqq t \leqq b$,

(3) $$\int_C f(z)\, dz = \int_a^b f(z(t))\dot{z}(t)\, dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Cauchy's integral theorem is the most important theorem in this chapter. It states that if $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D (Sec. 14.2),

(4) $$\oint_C f(z)\, dz = 0.$$

EXAMPLE 4 **Ratio Test**

Is the following series convergent or divergent? (First guess, then calculate.)

$$\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!} = 1 + (100+75i) + \frac{1}{2!}(100+75i)^2 + \cdots$$

Solution. By Theorem 8, the series is convergent, since

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+75i|^{n+1}/(n+1)!}{|100+75i|^n/n!} = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \to L = 0. \qquad \blacksquare$$

EXAMPLE 5 **Theorem 7 More General Than Theorem 8**

Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$. Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

Solution. The ratios of the absolute values of successive terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$. Hence convergence follows from Theorem 7. Since the sequence of these ratios has no limit, Theorem 8 is not applicable. $\blacksquare$

Root Test

The ratio test and the root test are the two practically most important tests. The ratio test is usually simpler, but the root test is somewhat more general.

THEOREM 9 **Root Test**

If a series $z_1 + z_2 + \cdots$ is such that for every n greater than some N,

$$\sqrt[n]{|z_n|} \leqq q < 1 \qquad (n > N) \tag{9}$$

(where $q < 1$ is fixed), this series converges absolutely. If for infinitely many n,

$$\sqrt[n]{|z_n|} \geqq 1, \tag{10}$$

the series diverges.

PROOF If (9) holds, then $|z_n| \leqq q^n < 1$ for all $n > N$. Hence the series $|z_1| + |z_2| + \cdots$ converges by comparison with the geometric series, so that the series $z_1 + z_2 + \cdots$ converges absolutely. If (10) holds, then $|z_n| \geqq 1$ for infinitely many n. Divergence of $z_1 + z_2 + \cdots$ now follows from Theorem 3. $\blacksquare$

CAUTION! Equation (9) implies $\sqrt[n]{|z_n|} < 1$, but this does not imply convergence, as we see from the harmonic series, which satisfies $\sqrt[n]{1/n} < 1$ (for $n > 1$) but diverges.

If the sequence of the roots in (9) and (10) converges, we more conveniently have

THEOREM 10 **Root Test**

If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then:

(a) *The series converges absolutely if $L < 1$.*

(b) *The series diverges if $L > 1$.*

(c) *If $L = 1$, the test fails; that is, no conclusion is possible.*

PROBLEM SET 15.1

1–10 SEQUENCES

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

1. $z_n = (1+i)^{2n}/2^n$
2. $z_n = (1+2i)^n/n!$
3. $z_n = n\pi/(2+4ni)$
4. $z_n = (2-i)^n$
5. $z_n = (-1)^n + 5i$
6. $z_n = (\cos 2n\pi i)/n$
7. $z_n = n^2 - i/2n^2$
8. $z_n = [(1+2i)/\sqrt{5}]^n$
9. $z_n = (2+2i)^{-n}$
10. $z_n = \sin(\frac{1}{4}n\pi) + i^n$
11. **CAS EXPERIMENT. Sequences.** Write a program for graphing complex sequences. Use the program to discover sequences that have interesting "geometric" properties, e.g., lying on an ellipse, spiraling to its limit, having infinitely many limit points, etc.
12. **Addition of sequences.** If $z_1, z_2, \cdots$ converges with the limit l and $z_1^*, z_2^*, \cdots$ converges with the limit l^*, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ is convergent with the limit $l + l^*$.
13. **Bounded sequence.** Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.
14. **On Theorem 1.** Illustrate Theorem 1 by an example of your own.
15. **On Theorem 2.** Give another example illustrating Theorem 2.

16–25 SERIES

Is the given series convergent or divergent? Give a reason. Show details.

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$
17. $\displaystyle\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$
18. $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{i}{4}\right)^n$
19. $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$
20. $\displaystyle\sum_{n=0}^{\infty} \frac{n+i}{3n^2+2i}$
21. $\displaystyle\sum_{n=0}^{\infty} \frac{(\pi+\pi i)^{2n+1}}{(2n+1)!}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$
23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^{2n}}{(2n)!}$
24. $\displaystyle\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$
25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
26. **Significance of (7).** What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?
27. **On Theorems 7 and 8.** Give another example showing that Theorem 7 is more general than Theorem 8.
28. **CAS EXPERIMENT. Series.** Write a program for computing and graphing numeric values of the first n partial sums of a series of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.
29. **Absolute convergence.** Show that if a series converges absolutely, it is convergent.
30. **Estimate of remainder.** Let $|z_{n+1}/z_n| \leqq q < 1$, so that the series $z_1 + z_2 + \cdots$ converges by the ratio test. Show that the remainder $R_n = z_{n+1} + z_{n+2} + \cdots$ satisfies the inequality $|R_n| \leqq |z_{n+1}|/(1-q)$. Using this, find how many terms suffice for computing the sum s of the series

$$\sum_{n=1}^{\infty} \frac{n+i}{2^n n}$$

with an error not exceeding 0.05 and compute s to this accuracy.

XAMPLE 6 Extension of Theorem 2

Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4+\cdots.$$

Solution. The sequence of the ratios $\frac{1}{6}$, $2(2+\frac{1}{4})$, $1/(8(2+\frac{1}{4}))$, $\cdots$ does not converge, so that Theorem 2 is of no help. It can be shown that

$$R = 1/\tilde{L}, \qquad \tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|}. \tag{6*}$$

This still does not help here, since $(\sqrt[n]{|a_n|})$ does not converge because $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = \frac{1}{2}$ for odd n, whereas for even n we have

$$\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1 \quad \text{as} \quad n\to\infty,$$

so that $\sqrt[n]{|a_n|}$ has the two limit points $\frac{1}{2}$ and 1. It can further be shown that

$$R = 1/\tilde{l}, \qquad \tilde{l} \text{ the greatest limit point of the sequence } \left\{\sqrt[n]{|a_n|}\right\}. \tag{6**}$$

Here $\tilde{l} = l$, so that $R = 1$. *Answer.* The series converges for $|z| < 1$. ■

Summary. Power series converge in an open circular disk or some even for every z (or some only at the center, but they are useless); for the radius of convergence, see (6) or Example 6.

Except for the useless ones, power series have sums that are analytic functions (as we show in the next section); this accounts for their importance in complex analysis.

ROBLEM SET 15.2

Power series. Are $1/z + z + z^2 + \cdots$ and $z + z^{3/2} + z^2 + z^3 + \cdots$ power series? Explain.

Radius of convergence. What is it? Its role? What motivates its name? How can you find it?

Convergence. What are the only basically different possibilities for the convergence of a power series?

On Examples 1–3. Extend them to power series in powers of $z - 4 + 3\pi i$. Extend Example 1 to the case of radius of convergence 6.

Powers z^{2n}. Show that if $\Sigma a_n z^n$ has radius of convergence R (assumed finite), then $\Sigma a_n z^{2n}$ has radius of convergence $\sqrt{R}$.

18 RADIUS OF CONVERGENCE

d the center and the radius of convergence.

$\sum_{n=0}^{\infty} 2^n(z-1)^n$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z-\frac{1}{4}\pi\right)^{2n}$

8. $\sum_{n=0}^{\infty} \frac{n^n}{n!}(z-\pi i)^n$

9. $\sum_{n=0}^{\infty} \frac{n(n-1)}{2^n}(z+i)^{2n}$

10. $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$

11. $\sum_{n=0}^{\infty} \left(\frac{3-i}{5+2i}\right)z^n$

12. $\sum_{n=0}^{\infty} \frac{(-1)^n n}{8^n} z^n$

13. $\sum_{n=0}^{\infty} 16^n(z+i)^{4n}$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}(n!)^2} z^{2n}$

15. $\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2}(z-2i)^n$

16. $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n(n!)^3} z^n$

17. $\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^{2n+1}$

18. $\sum_{n=0}^{\infty} \frac{2(-1)^n}{\sqrt{\pi}(2n+1)n!} z^{2n+1}$

19. **CAS PROJECT. Radius of Convergence.** Write a program for computing R from (6), (6*), or (6**), in this order, depending on the existence of the limits needed. Test the program on some series of your choice such that all three formulas (6), (6*), and (6**) will come up.

20. **TEAM PROJECT. Radius of Convergence.**

(a) **Understanding (6).** Formula (6) for R contains $|a_n/a_{n+1}|$, not $|a_{n+1}/a_n|$. How could you memorize this by using a qualitative argument?

(b) **Change of coefficients.** What happens to R $(0 < R < \infty)$ if you (i) multiply all a_n by $k \neq 0$, (ii) multiply all a_n by $k^n \neq 0$, (iii) replace a_n by $1/a_n$? Can you think of an application of this?

(c) **Understanding Example 6,** which extends Theorem 2 to nonconvergent cases of a_n/a_{n+1}. Do you understand the principle of "mixing" by which Example 6 was obtained? Make up further examples.

(d) **Understanding (b) and (c) in Theorem 1.** Does there exist a power series in powers of z that converges at $z = 30 + 10i$ and diverges at $z = 31 - 6i$? Give reason.

15.3 Functions Given by Power Series

Here, our main goal is to show that power series represent analytic functions. This fact (Theorem 5) and the fact that power series behave nicely under addition, multiplication, differentiation, and integration accounts for their usefulness.

To simplify the formulas in this section, we take $z_0 = 0$ and write

$$\sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

There is no loss of generality because a series in powers of $\hat{z} - z_0$ with any z_0 can always be reduced to the form (1) if we set $\hat{z} - z_0 = z$.

Terminology and Notation. If any given power series (1) has a nonzero radius of convergence R (thus $R > 0$), its sum is a function of z, say $f(z)$. Then we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R). \tag{2}$$

We say that $f(z)$ *is* **represented** *by the power series* or that *it is* **developed** *in the power series.* For instance, the geometric series *represents* the function $f(z) = 1/(1-z)$ in the interior of the unit circle $|z| = 1$. (See Theorem 6 in Sec. 15.1.)

Uniqueness of a Power Series Representation. This is our next goal. It means that *a function $f(z)$ cannot be represented by two different power series with the same center.* We claim that if $f(z)$ can at all be developed in a power series with center z_0, the development is unique. This important fact is frequently used in complex analysis (as well as in calculus). We shall prove it in Theorem 2. The proof will follow from

THEOREM 1 Continuity of the Sum of a Power Series

If a function $f(z)$ can be represented by a power series (2) with radius of convergence $R > 0$, then $f(z)$ is continuous at $z = 0$.