To get rid of u_y, multiply (6a) by u and (6b) by v and add. Similarly, to eliminate u_x, multiply (6a) by $-v$ and (6b) by u and add. This yields

$$(u^2 + v^2)u_x = 0,$$
$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $v_x = v_y = 0$. Together this implies $u = \text{const}$ and $v = \text{const}$; hence $f = \text{const}$. ■

We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are (Prob. 1)

$$(7) \qquad u_r = \frac{1}{r}v_\theta, \qquad v_r = -\frac{1}{r}u_\theta \qquad (r > 0).$$

Laplace's Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace's equation, the most important PDE of physics. It occurs in gravitation, electrostatics, fluid flow, heat conduction, and other applications (see Chaps. 12 and 18).

M 3 **Laplace's Equation**

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$(8) \qquad \nabla^2 u = u_{xx} + u_{yy} = 0$$

(∇^2 read "nabla squared") *and*

$$(9) \qquad \nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D and have continuous second partial derivatives in D.

OF Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we have

$$(10) \qquad u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}.$$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$. By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$. ■

Solutions of Laplace's equation having ***continuous*** second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory** (see also Sec. 12.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE 4 How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$ ■

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens a wide range of engineering and physical applications, as shown in Chap. 18.

PROBLEM SET 13.4

1. **Cauchy–Riemann equations in polar form.** Derive (7) from (1).

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$
3. $f(z) = e^{-x}\cos(y) - ie^{-x}\sin(y)$
4. $f(z) = e^{x}(\cos y - i\sin y)$
5. $f(z) = \text{Re}(z^2) - i\,\text{Im}(z^2)$
6. $f(z) = 1/(z - z^5)$
7. $f(z) = -i/z^4$
8. $f(z) = \text{Arg}\, z$
9. $f(z) = 3\pi^2/(z^3 + 4\pi^2 z)$
10. $f(z) = \ln|z| + i\,\text{Arg}\, z$
11. $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

12–19 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = x^3 + y^3$
13. $u = -2xy$
14. $v = xy$
15. $u = -\dfrac{x}{x^2 + y^2}$
16. $u = \sin x \cosh y$
17. $v = (2x - 1)y$
18. $u = x^3 - 3xy^2$
19. $v = e^{-x}\sin 2y$
20. **Laplace's equation.** Give the details of the derivative of (9).

21–24 Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{-\pi x}\cos ay$
22. $u = \cos ax \cosh 2y$
23. $u = ax^3 + bxy$
24. $u = \cosh ax \cos y$
25. **CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = \text{const}$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to **(a)** $u = x^2 - y^2$, $v = 2xy$, **(b)** $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$.
26. Apply the program in Prob. 25 to $u = e^x \cos y$, $v = e^x \sin y$ and to an example of your own.

Periodicity of e^x with period $2\pi i$,

(12) $$e^{z+2\pi i} = e^z \qquad \text{for all } z$$

is a basic property that follows from (1) and the periodicity of cos y and sin y. Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π

(13) $$-\pi < y \leqq \pi \qquad \text{(Fig. 336).}$$

This infinite strip is called a **fundamental region** of e^z.

[EXAM]PLE 1 Function Values. Solution of Equations

Computation of values from (1) provides no problem. For instance,

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$

$$|e^{1.4-1.6i}| = e^{1.4} = 4.055, \qquad \text{Arg } e^{1.4-0.6i} = -0.6.$$

To illustrate (3), take the product of

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i \sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ■

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

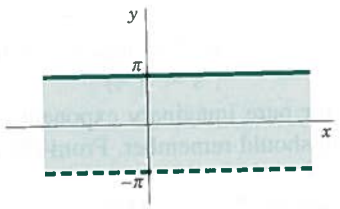

Fig. 336. Fundamental region of the exponential function e^z in the z-plane

[PR]OBLEM SET 13.5

[...]s **entire.** Prove this.

Function Values. Find e^z in the form $u + iv$ if z equals

[...] $4i$ **3.** $2\pi i(1 - i)$

[...] $- 1.8i$ **5.** $1 - 3\pi i$

[...] $\pi i/2$ **7.** $\sqrt{3} - \frac{\pi}{2}i$

8–13 Polar Form. Write in exponential form (6):

8. $\sqrt[n]{z}$ **9.** $3 - 4i$

10. $\sqrt{i}$, $\sqrt{-i}$ **11.** $-\frac{3}{2}$

12. $1/(1 - z)$ **13.** $1 - i$

14–17 Real and Imaginary Parts. Find Re and Im of

14. $e^{-\pi z}$ **15.** $\exp(-z^2)$

16. $e^{1/z}$ **17.** $\exp(z^3)$

18. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

19–22 Equations. Find all solutions and graph some of them in the complex plane.

19. $e^z = 1$ **20.** $e^z = 4 + 3i$

21. $e^z = 0$ **22.** $e^z = -2$

13.6 Trigonometric and Hyperbolic Functions. Euler's Formula

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions.* We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

(1) $$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

(2) $$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

and

(3) $$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Since e^z is entire, cos z and sin z are entire functions. tan z and sec z are not entire; they are analytic except at the points where cos z is zero; and cot z and csc z are analytic except

ROBLEM SET 13.6

4 **FORMULAS FOR HYPERBOLIC FUNCTIONS**
ıw that

$\cosh z = \cosh x \cos y + i \sinh x \sin y$

$\sinh z = \sinh x \cos y + i \cosh x \sin y.$

$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$

$\cosh^2 z - \sinh^2 z = 1, \quad \cosh^2 z + \sinh^2 z = \cosh 2z$

Entire Functions. Prove that $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ are entire.

Harmonic Functions. Verify by differentiation that $\text{Im} \cos z$ and $\text{Re} \sin z$ are harmonic.

-12 **Function Values.** Find, in the form $u + iv$,

. $\sin \frac{\pi}{2} i$

7. $\cos(-i)$, $\sin(-i)$

. $\cos \pi i$, $\cosh \pi i$

. $\cosh(-2 + i)$, $\cos(-1 - 2i)$

. $\sinh(3 + 4i)$, $\cosh(3 + 4i)$

11. $\sin \frac{\pi}{4} i$, $\cos(\frac{\pi}{2} - \frac{\pi}{4} i)$

12. $\cos \frac{1}{2}\pi i$, $\cos[\frac{1}{2}\pi(1 + i)]$

13–15 **Equations and Inequalities.** Using the definitions, prove:

13. $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.

14. $|\sinh y| \leqq |\cos z| \leqq \cosh y, |\sinh y| \leqq |\sin z| \leqq \cosh y.$ Conclude that the complex cosine and sine are not bounded in the whole complex plane.

15. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin(z_1 + z_2) + \sin(z_1 - z_2)]$

16–19 **Equations.** Find all solutions.

16. $\sin z = 100$

17. $\cosh 2z = 0$

18. $\cosh z = -1$

19. $\sinh z = 0$

20. **Re tan z and Im tan z.** Show that

$$\text{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\text{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

3.7 Logarithm. General Power. Principal Value

We finally introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now, from Sec. 13.5, we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \quad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z). \tag{1}$$

Now comes an important point (without analog in real calculus). Since the argument of z is determined only up to integer multiples of 2π, ***the complex natural logarithm*** $\ln z$ ***($z \neq 0$) is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\text{Arg}\, z$ (see Sec. 13.2) is denoted by $\text{Ln}\, z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\text{Ln}\, z = \ln |z| + i \,\text{Arg}\, z \qquad (z \neq 0). \tag{2}$$

The uniqueness of $\text{Arg}\, z$ for given z ($\neq 0$) implies that $\text{Ln}\, z$ is single-valued, that is, a function in the usual sense. Since the other values of $\arg z$ differ by integer multiples of 2π, the other values of $\ln z$ are given by

$$\ln z = \text{Ln}\, z \pm 2n\pi i \qquad (n = 1, 2, \cdots). \tag{3}$$

They all have the same real part, and their imaginary parts differ by integer multiples of 2π.

If z is positive real, then $\text{Arg}\, z = 0$, and $\text{Ln}\, z$ becomes identical with the real natural logarithm known from calculus. If z is negative real (so that the natural logarithm of calculus is not defined!), then $\text{Arg}\, z = \pi$ and

$$\text{Ln}\, z = \ln |z| + \pi i \qquad (z \text{ negative real}).$$

From (1) and $e^{\ln r} = r$ for positive real r we obtain

$$e^{\ln z} = z \tag{4a}$$

as expected, but since $\arg(e^z) = y \pm 2n\pi$ is multivalued, so is

$$\ln(e^z) = z \pm 2n\pi i, \qquad n = 0, 1, \cdots. \tag{4b}$$

EXAMPLE 1 Natural Logarithm. Principal Value

$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \cdots$	$\text{Ln}\, 1 = 0$
$\ln 4 = 1.386294 \pm 2n\pi i$	$\text{Ln}\, 4 = 1.386294$
$\ln(-1) = \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots$	$\text{Ln}(-1) = \pi i$
$\ln(-4) = 1.386294 \pm (2n + 1)\pi i$	$\text{Ln}(-4) = 1.386294 + \pi i$
$\ln i = \pi i/2, -3\pi i/2, 5\pi i/2, \cdots$	$\text{Ln}\, i = \pi i/2$
$\ln 4i = 1.386294 + \pi i/2 \pm 2n\pi i$	$\text{Ln}\, 4i = 1.386294 + \pi i/2$
$\ln(-4i) = 1.386294 - \pi i/2 \pm 2n\pi i$	$\text{Ln}(-4i) = 1.386294 - \pi i/2$
$\ln(3 - 4i) = \ln 5 + i \arg(3 - 4i)$	$\text{Ln}(3 - 4i) = 1.609438 - 0.927295i$
$= 1.609438 - 0.927295i \pm 2n\pi i$	(Fig. 337) ■

It is a *convention* that for real positive $z = x$ the expression z^c means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\text{Ln } z$ $(z = x > 0)$ in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a} \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

PROBLEM SET 13.7

1–4 VERIFICATIONS IN THE TEXT

1. Verify the computations in Example 1.
2. Verify (5) for $z_1 = -i$ and $z_2 = -1$.
3. Prove analyticity of $\text{Ln } z$ by means of the Cauchy–Riemann equations in polar form (Sec. 13.4).
4. Prove (4a) and (4b).

COMPLEX NATURAL LOGARITHM *ln z*

5–11 Principal Value Ln *z*. Find $\text{Ln } z$ when z equals

5. -7
6. $8 + 8i$
7. $8 - 8i$
8. $1 \pm i$
9. $0.6 - 0.8i$
10. $-15 \pm 0.1i$
11. $-ei^2$

12–16 All Values of ln *z*. Find all values and graph some of them in the complex plane.

12. $\ln e$
13. $\ln 1$
14. $\ln(-5)$
15. $\ln(e^i)$
16. $\ln(4 - 3i)$
17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z.

18. $\ln z = \pi i/2$
19. $\ln z = 4 - 3i$
20. $\ln z = e + \pi i$
21. $\ln z = 0.4 + 0.2i$

22–28 General Powers. Find the principal value. Show details.

22. $(2i)^{2i}$
23. $(1 + i)^{1-i}$
24. $(1 - i)^{1+i}$
25. $(-3)^{3-i}$
26. $(i)^{i/2}$
27. $(-1)^{2-i}$
28. $(3 + 4i)^{1/3}$
29. How can you find the answer to Prob. 24 from the answer to Prob. 23?
30. **TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine**, etc., are defined and denoted in a similar fashion. (Note that all these relations are *multivalued*.) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

    (a) $\arccos z = -i \ln(z + \sqrt{z^2 - 1})$

    (b) $\arcsin z = -i \ln(iz + \sqrt{1 - z^2})$

    (c) $\text{arccosh } z = \ln(z + \sqrt{z^2 - 1})$

    (d) $\text{arcsinh } z = \ln(z + \sqrt{z^2 + 1})$

    (e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

    (f) $\text{arctanh } z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

    (g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

1. Divide $4 + 7i$ by $-1 + 2i$. Check the result by multiplication.
2. What happens to a quotient if you take the complex conjugates of the two numbers? If you take the absolute values of the numbers?
3. Write the two numbers in Prob. 1 in polar form. Find the principal values of their arguments.
4. State the definition of the derivative from memory. Explain the big difference from that in calculus.
5. What is an analytic function of a complex variable?
6. Can a function be differentiable at a point without being analytic there? If yes, give an example.
7. State the Cauchy–Riemann equations. Why are they of basic importance?
8. Discuss how e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are related.
9. $\ln z$ is more complicated than $\ln x$. Explain. Give examples.
10. How are general powers defined? Give an example. Convert it to the form $x + iy$.

11–16 Complex Numbers. Find, in the form $x + iy$, showing details,

11. $(4 + 5i)^2$
12. $(1 - i)^{10}$
13. $1/(3 - 4i)$
14. $\sqrt{i}$
15. $(1 - i)/(1 + i)$
16. $e^{\pi i/2}$, $e^{-\pi i/2}$

17–20 Polar Form. Represent in polar form, with the principal argument.

17. $2 - 2i$
18. $12 + i$, $12 - i$
19. $-5i$
20. $0.6 + 0.8i$

21–24 Roots. Find and graph all values of:

21. $\sqrt[4]{625}$
22. $\sqrt{-32i}$
23. $\sqrt[4]{-1}$
24. $\sqrt[3]{1}$

25–30 Analytic Functions. Find $f(z) = u(x, y) + iv(x, y)$ with u or v as given. Check by the Cauchy–Riemann equations for analyticity.

25 $u = -xy$
26. $v = y/(x^2 + y^2)$
27. $v = -e^{-3x} \sin 3y$
28. $u = \cos 3x \cosh 3y$
29. $u = \exp(-(x^2 - y^2)/2) \cos xy$
30. $v = \cos 2x \sinh 2y$

31–35 Special Function Values. Find the value of:

31. $\cos(5 - 2i)$
32. $\text{Ln}(0.6 + 0.8i)$
33. $\tan(1 + i)$
34. $\sinh(1 + \pi i)$, $\sin(1 + \pi i)$
35. $\sinh(\pi - \pi i)$

SUMMARY OF CHAPTER 13
Complex Numbers and Functions. Complex Differentiation

For arithmetic operations with **complex numbers**

$$z = x + iy = re^{i\theta} = r(\cos\theta + i \sin\theta), \tag{1}$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{2}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).