15.4-5 Taylor series and uniform convergence
16.2 Zeros of analytic functions

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Expansions of analytic functions into power series

$z_0 \in \mathbb{C}$, $R > 0$, $f(z)$ is analytic in $\{z : |z - z_0| < R\}$.

Cauchy representation + expansion of the Cauchy kernel $\Rightarrow$

$$f(z) = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta =$$

$$\sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

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$$\sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta}_{\frac{1}{n!} f^{(n)}(z_0)} =$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0).$$

We obtained Taylor series expansion for analytic function. It converges for all z such that $|z - z_0| < R$.

Taylor's formula with remainder

If f is analytic in a disk $\{|z - z_0| < R\}$ then f can be represented by a power series

$$f(z) = \sum_{n=}^{\infty} a_n z^n$$

in this disk and $a_n = \frac{f^{(n)}(z_0)}{n!}$. We call it the Taylor series of f.

Taking partial sums of the Taylor series we obtain polynomial approximation of an analytic function

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + ... + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z)$$

The remainder is given by the formula

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_{|\zeta - z_0| = R} \frac{f(\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)} d\zeta$$

Taylor series as power series

Theorem

Let $f(z)$ be analytic in a domain D and z_0 be any point in D. There there is precisely one Taylor series with center z_0 that represents $f(z)$. The disk of convergence for this series is the largest disk centered at z_0 where $f(z)$ is analytic.

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Example

1. $f(z) = \frac{1}{1+z^2}$, $z_0 = 0$ the Taylor series converges for $|z| < 1$
2. $f(z) = (\cos z)^{-1}$, $z_0 = 0$ the Taylor series converges for $|z| < \pi/2$.

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A power series with a nonzero radius of convergence is the Taylor series of its sum.

Convergence of power series

Theorem

Suppose that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *converges at some point* $z_1 \neq z_0$. *Then it converges for each* z *such that* $|z - z_0| \leq r < |z_1 - z_0|$.

We note that the sequence $a_n|z_1 - z_0|^n$ is bounded, since the series converges. Thus for some constant M we have $|a_n| \leq Mr_1^{-n}$. Then for $|z - z_0| \leq r$ we have

$$|a_n(z - z_0)^n| \leq M(r/r_1)^n$$

The series converges by the comparison principle.

Uniform convergence (optional)

Definition

We say that a series of functions $\sum_{n=0}^{\infty} g_n(z)$ converges uniformly on some set D if for each z_0 in D the series $\sum_{n=0}^{\infty} g_n(z_0)$ converges and for each $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\left| \sum_{n=N_1}^{N_2} g_n(z) \right| < \epsilon, \quad \text{if } N_1, N_2 > N, \text{ and } z \in D$$

Our proof above shows that if a power series converges at some point z_1 then it converges uniformly in a disk $|z - z_0| \leq r < r_1$.

Even and odd functions

Suppose that $f(z)$ is analytic at some domain D that contains the origin. We say that $f(z)$ is even if for each z such that z and $-z$ are in D we have $f(-z) = f(z)$.
Let $D_- = \{-z : z \in D\}$. Then an even function f can be extended to an analytic function in $D \cup D_-$. The Taylor series of an even function centered at the origin contains only the even powers:

$$f(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}$$

Similarly, the Taylor series of an odd function $g(z)$, $g(-z) = -g(z)$, contains only odd powers of z.

Examples

Example

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$, it is an odd function
- $\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$, it is an even function
- We consider functions $f_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}$, $f_2(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k$, $f_3(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^k$.

  All of them are analytic at the whole complex plane, and $f_1(z) = \frac{\sin z}{z}$, is an even function, $f_2(z) = \cos\sqrt{z}$ and $f_3(z) = f_1(\sqrt{z})$.

Uniqueness theorem for analytic functions

Theorem

Let f be analytic in a domain D and z_0 be a point in D, if $f^{(n)}(z_0) = 0$ for each n then $f(z) = 0$ for all z in D

First we know that the Taylor series of f at z_0 converges to f in some dick centered at z_0. This means that $f = 0$ at all points of such disk. We can now take zny other point z_1 in D connect z and z_1 by a curve and take a sequences of disks. The first one B_1 centered at $w_1 = z_0$ and contained in D. Second B_2 is centered on a point w_2 of the curve inside such that w_2 is inside B_1 and so on, we will connect z_0 and z_1 by a chain of such balls and show that in each ball $f = 0$. Then $f(z_1) = 0$.

Corollary

If f and g are to analytic functions in D and $f^{(n)}(z_0) = g^{(n)}(z_0)$ at some point then $f = g$.

Zeros of analytic functions

Definition

Let f be an analytic function in $\{|z - z_0| < r\}$ we say that f has zero of order d at z_0 if the first non-zero term in the Fourier series of f centered at z_0 is $a_d(z - z_0)^d$.

f has zero of order d if and only if $f(z_0) = f'(z_0) = ... = f^{(d-1)}(z_0) = 0$ and $f^{(d)}(z_0) \neq 0$. In this case we have

$$f(z) = \sum_{n=d}^{\infty} a_n(z - z_0)^n = (z - z_0)^d \sum_{n=0}^{\infty} a_{n+d}(z - z_0)^n = (z - z_0)^d g(z),$$

where $g(z)$ is given by a convergent power series and g is analytic in $\{|z - z_0| < r\}$, moreover $g(z_0) = a_d \neq 0$

Zeros

Isolated zeros: If f is analytic in $\{|z - z_0| < r\}$ and $f(z_0) = 0$ but f is not identically zero then there exists some $\epsilon > 0$ such that $f(z) \neq 0$ when $0 < |z - z_0 < \epsilon$. We say that the zeros of analytic functions are isolated.

Ratios of two analytic functions: If $f_1(z)$ and $f_2(z)$ are two analytic functions with zeros of order d_1 and d_2 at z_0 and $d_2 \leq d_1$. Then the ratio f_1/f_2 can be extended to z_0 such that the resulting function is analytic in some disk $|z - z_0| < \epsilon$.
We write $f_1(z) = (z - z_0)^{d_1} g_1(z)$ and $f_2(z) = (z - z_0)^{d_2} g_2(z)$ and get

$$\frac{f_1(z)}{f_2(z)} = (z - z_0)^{d_1 - d_2} \frac{g_1(z)}{g_2(z)}$$

where g_1 and g_2 have no zeros in some disk centered at z_0.