



## **15.2-3 Complex power series.**

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## Power series

We will study functions represented by power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

$z_0$  - center of the power series.

- The series converges in a disk centered at  $z_0$ ,  $\{z : |z - z_0| < R\}$ ,  $0 \leq R \leq +\infty$  (this disk may degenerate to a single point  $z_0$  or be the whole plane)
- The series diverges outside the closed disk, i.e. it diverges when  $|z - z_0| > R$ . We don't know the behavior on the circle  $\{z : |z - z_0| = R\}$ , it depends on the coefficients.
- Formula for the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} \text{ if the limit exists,}$$

## Examples

1.  $\sum_{n=0}^{\infty} \frac{z^n}{n!}$  converges everywhere,  $R = \infty$ , as for real numbers we have  $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$ .
2.  $\sum_{n=0}^{\infty} z^n$  converges when  $|z| < 1$  to the sum  $\frac{1}{1-z}$  and diverges when  $|z| \geq 1$ ,  $R = 1$ .
3.  $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$  converges when  $|z| \leq 1$  and diverges when  $|z| > 1$ ,  $R = 1$
4.  $\sum_{n=1}^{\infty} \frac{z^n}{n}$  converges when  $|z| < 1$  and diverges when  $|z| > 1$ ,  $R = 1$ ; when  $z = 1$  the series diverges, when  $z = -1$  it converges. (What happens for other  $z$  on the unit circle?)
5.  $\sum_{n=0}^{\infty} n! z^n$  diverges when  $z \neq 0$ ,  $R = 0$  (use the formula for  $R$ ).
6.  $\sum_{n=0}^{\infty} 2^n (z+i)^{2n}$  converges when  $|z+i| < 1/\sqrt{2}$ ,  $R = 1/\sqrt{2}$ .
7.  $\sum_{n=0}^{\infty} n(z-1)^n$  converges when  $|z-1| < 1$ ,  $R = 1$ .

## The sum of power series



Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a power series with positive radius of convergence  $R$ . Then for each  $z : |z - z_0| < R$  the sum

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is defined. We say that  $f$  is represented by the power series.

### Theorem

*If  $f$  is represented by a power series in some disk  $\{z : |z - z_0| < R\}$  with  $R > 0$  then  $f$  is continuous at  $z_0$ .*

## Uniqueness of power series and operation

### Theorem

Suppose that  $f(z)$  is represented by two series  $\sum a_n(z - z_0)^n$  and  $\sum_n b_n(z - z_0)^n$  in some disk centered at  $z_0$ . Then these series are equal,  $a_n = b_n$ .

Let  $f(z) = \sum a_n(z - z_0)^n$  and  $g(z) = \sum_n b_n(z - z_0)^n$  and both radii of convergence are  $\geq R$ . Then

- The series  $\sum_n (a_n + b_n)(z - z_0)^n$  converges in  $\{|z - z_0| < R\}$  to  $f(z) + g(z)$
- For  $\{z : |z - z_0| < R\}$  the product  $f(z)g(z)$  is represented by the series

$$\begin{aligned} a_0 b_0 + (a_0 b_1 + b_0 a_1)(z - z_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(z - z_0)^2 + \dots = \\ = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) z^n \end{aligned}$$

## Term-wise differentiation and integration



Consider the following power series

1.  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$
2.  $\sum_{n=0}^{\infty} n a_n(z - z_0)^{n-1}$
3.  $\sum_{n=1}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$

These series has the same radius of convergence. The second one is obtained from the first by term-wise differentiation. The third is the term-wise integration of the first.

### Theorem

*If  $f(z)$  is represented by a power series  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  with a positive radius of convergence  $R$ , then  $f(z)$  is analytic in the open disk  $\{z : |z - z_0| < R\}$  and  $f'(z)$  is represented by the series  $\sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$ .*

## Term-wise integration



### Theorem

If  $f(z)$  is represented by a power series  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  with a positive radius of convergence  $R$ , then the anti-derivative  $F(z)$  is represented by the series  $C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$ .

### Example

If  $f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$  then

$$\log(1 - z) = - \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1$$

## Examples

1.  $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$  and  $(e^z)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = e^z$ .
2.  $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$  and  $(1-z)^{-2} = \sum_{n=1}^{\infty} nz^n$ .
3.  $\sum_{n=1}^{\infty} \frac{(z-i)^n}{n3^n}$  converges to an analytic function  $f$  in  $|z-i| < 3$  and  $f'(z) = \sum_{n=1}^{\infty} \frac{(z-i)^{n-1}}{3^n} = \frac{1}{3(1-(z-i)/3)} = \frac{3}{3+i-z}$ . Then
$$f(z) = -3 \ln(z - 3 + i)$$

when  $|z-i| < 3$

4.  $\sum_{n=0}^{\infty} n(n-1)(z-\pi i)^n$ , the radius of convergence is  $R=1$  and

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^n = (z-\pi i)^2 \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^{n-2} \\ &= (z-\pi i)^2 \left( \sum_{n=0}^{\infty} (z-\pi i)^n \right)'' = (z-\pi i)^2 \left( \frac{1}{1+\pi i-z} \right)'' = \\ &= \frac{2(z-\pi i)^2}{(1+\pi i-z)^3} \end{aligned}$$



## Power series: example

Geometric series:  $z \in \mathbb{C}$  and  $w_n = z^n$

$$\sum_0^{\infty} z^n = \begin{cases} \frac{1}{1-z}, & |z| < 1, \\ \text{diverges}, & |z| \geq 1. \end{cases}$$

Expansion of the Cauchy kernel:

Fix  $z_0 \in \mathbb{C}$  and let  $\zeta, z \in \mathbb{C}$  be such that  $|z - z_0| < |\zeta - z_0|$ . Then

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \\ &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \end{aligned}$$

We are going to use this formula for making expansions of analytic functions into power series !