14.4 Derivatives of analytic functions
15.1 Complex series

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Derivatives of analytic functions

**Theorem**

If $f$ is analytic in some domain $D$ then it has derivatives of any order which are also analytic functions. Moreover,

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz,
\]

\[
f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz,
\]

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz,
\]

where $C$ is a simple closed path in $D$ that bounds some domain in $D$ which contains $z_0$. 
Examples

\[ \oint_C \frac{z}{(z-z_0)^2} dz = 2\pi i \]

\[ \oint_C \frac{e^z}{(z-z_0)^4} dz = \frac{2\pi i e^{z_0}}{6}, \quad (e^z)''' = e^z \]

\[ \oint_C \frac{\cos z}{(z-\pi)^3} dz = \frac{-2\pi i \cos \pi}{24} = \frac{\pi i}{12}, \quad (\cos z)'' = -\cos z \]
Cauchy’s inequality

Suppose that $f$ is an analytic function in a disc of radius $r$ around $z_0$ and that $|f(z)| \leq M$ when $|z - z_0| = r$. Then

$$|f(z)| \leq M, \quad \text{and} \quad |f^{(n)}(z_0)| \leq \frac{n!M}{r^n}, \quad \text{for } |z - z_0| \leq r.$$ 

Let $C = \{z : |z - z_0| = r\}$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

then taking the absolute values and applying $ML$-inequality, we get

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!M}{r^n}$$
Applications

Theorem (Liouville)

If a function is analytic in a whole complex plane and bounded in absolute value, then it is a constant.

Functions analytic in the whole plane are called entire functions.

Theorem (Morera)

If $f$ is continuous in a simply connected domain $D$ and

$$\oint_C f(z)dz = 0$$

for any closed curve $C$. Then $f$ is analytic in $D$. 
Series of complex numbers

Given a sequence \( \{ w_n \}_{n=1}^{\infty} \in \mathbb{C} \) consider the series \( \sum_{n=1}^{\infty} w_n \). The series is convergent if its partial sums have a limit:

\[
S_N = \sum_{n=1}^{N} w_n \to S, \text{ as } N \to \infty.
\]

This limit is called the sum of the series. Otherwise the series is divergent.

Relation to series of real numbers:
Let \( w_n = u_n + iv_n \). The series \( \sum w_n \) converges if and only if each of the real series \( \sum u_n, \sum v_n \) converges.
Basic definitions and facts

— The series $\sum w_n$ is absolutely convergent if $\sum |w_n|$ is convergent.
— The series $\sum w_n$ is convergent $\Rightarrow w_n \to 0$ as $n \to \infty$

Some sufficient conditions for convergence

— Cauchy criterion: $\sum_{M}^{N} w_n \to 0$ as $M, N \to \infty \iff \sum w_n$ converges
— Majorization: $a_n > 0, \ n = 1, 2, \ldots, |w_n| \leq a_n$ and $\sum a_n$ converges $\Rightarrow \sum w_n$ converges;
— Ratio test: $|w_{n+1}|/|w_n| \leq q < 1 \ n = 1, 2, \ldots \Rightarrow \sum w_n$ converges;
— Root test: $(|w_n|)^{1/n} \leq q < 1 \ n = 1, 2, \ldots \Rightarrow \sum w_n$ converges
The most important example

Geometric series: \( z \in \mathbb{C} \) and \( w_n = z^n \)

\[
\sum_{0}^{\infty} z^n = \begin{cases} 
\frac{1}{1-z}, & |z| < 1, \\
\text{diverges}, & |z| \geq 1.
\end{cases}
\]

Expansion of the Cauchy kernel:
Fix \( z_0 \in \mathbb{C} \) and let \( \zeta, z \in \mathbb{C} \) be such that \( |z - z_0| < |\zeta - z_0| \). Then

\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} =
\]

\[
\frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}
\]

We are going to use this formula for making expansions of analytic functions into power series!
Further examples

1. \[ \sum_{n=1}^{\infty} \frac{(1+i)^n}{n!} \] converges (ratio test)

2. \[ \sum_{n=1}^{\infty} \frac{(1+i)^{2n}}{2^n} \] diverges, \( |(1 + i)^{2n}/2^n| = 1 \not\to 0 \)

3. \[ \sum_{n=1}^{\infty} \frac{n+i}{n^2} \] diverges, the real parts are \(1/n\) and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges

4. \[ \sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}} \] converges,

\[ \sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}} = \frac{i}{1} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots \]

Separating real and imaginary parts we get two real alternating series, both of them converge and then the series converge. It does not converge absolutely, \( \sum_n \left| \frac{i^n}{\sqrt{n}} \right| \) diverges.