14.4 Derivatives of analytic functions
15.1 Complex series

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Derivatives of analytic functions

Theorem

If f is analytic in some domain D then it has derivatives of any order which are also analytic functions. Moreover,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz,$$

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz,$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a simple closed path in D that bounds some domain in D which contains z_0.

Examples

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$$\oint_C \frac{z}{(z-z_0)^2}dz = 2\pi i$$

—

$$\oint_C \frac{e^z}{(z-z_0)^4}dz = \frac{2\pi i e^{z_0}}{6}, \quad (e^z)''' = e^z$$

—

$$\oint_C \frac{\cos z}{(z-\pi)^3}dz = \frac{-2\pi i \cos\pi}{24} = \frac{\pi i}{12}, \quad (\cos z)'' = -\cos z$$

Cauchy's inequality

Suppose that f is an analytic function in a disc of radius r around z_0 and that $|f(z)| \leq M$ when $|z - z_0| = r$. Then

$$|f(z)| \leq M, \quad \text{and} \quad |f^{(n)}(z_0)| \leq \frac{n!M}{r^n}, \quad \text{for } |z - z_0| \leq r.$$

Let $C = \{z : |z - z_0| = r\}$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

then taking the absolute values and applying ML-inequality, we get

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

Applications

Theorem (Liouville)

If a function is analytic in a whole complex plane and bounded in absolute value, then it is a constant

Functions analytic in the whole plane are called entire functions.

Theorem (Morera)

If f is continuous in a simply connected domain D and

$$\oint_C f(z)dz = 0$$

for any closed curve C. Then f is analytic in D.

Series of complex numbers

Given a sequence $\{w_n\}_{n=1}^{\infty} \in \mathbb{C}$ consider *the series* $\sum_{n=1}^{\infty} w_n$ The series is convergent if its partial sums have a limit:

$$S_N = \sum_{n=1}^{N} w_n \ \to S, \text{ as } N \to \infty.$$

This limit is called the sum of the series.
Otherwise the series is divergent.

Relation to series of real numbers:
Let $w_n = u_n + iv_n$. The series $\sum w_n$ converges if and only if each of the real series $\sum u_n$, $\sum v_n$ converges.

Basic definitions and facts

- The series $\sum w_n$ is absolutely convergent if $\sum |w_n|$ is convergent.
- The series $\sum w_n$ is convergent $\Rightarrow w_n \to 0$ as $n \to \infty$

Some sufficient conditions for convergence

- Cauchy criterion: $\sum_M^N w_n \to 0$ as $M, N \to \infty \Leftrightarrow \sum w_n$ converges
- Majorization: $a_n > 0,\ n = 1, 2, \ldots, |w_n| \le a_n$ and $\sum a_n$ converges $\Rightarrow \sum w_n$ converges;
- Ratio test: $|w_{n+1}|/|w_n| \le q < 1\ n = 1, 2,\ \ldots \Rightarrow \sum w_n$ converges;
- Root test: $(|w_n|)^{1/n} \le q < 1\ n = 1, 2,\ \ldots \Rightarrow \sum w_n$ converges

The most important example

Geometric series: $z \in \mathbb{C}$ and $w_n = z^n$

$$\sum_0^\infty z^n = \begin{cases} \frac{1}{1-z}, & |z| < 1, \\ \text{diverges}, & |z| \geq 1. \end{cases}$$

Expansion of the Cauchy kernel:
Fix $z_0 \in \mathbb{C}$ and let $\zeta, z \in \mathbb{C}$ be such that $|z - z_0| < |\zeta - z_0|$. Then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} =$$

$$\frac{1}{\zeta - z_0} \sum_{n=0}^\infty \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

We are going to use this formula for making expansions of analytic functions into power series !

Further examples

1. $\sum_{n=1}^{\infty} \frac{(1+i)^n}{n!}$ converges (ratio test)
2. $\sum_{n=1}^{\infty} \frac{(1+i)^{2n}}{2^n}$ diverges , $|(1+i)^{2n}/2^n| = 1 \not\to 0$
3. $\sum_{n=1}^{\infty} \frac{n+i}{n^2}$ diverges, the real parts are $1/n$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
4. $\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}}$ converges,

$$\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}} = \frac{i}{1} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$$

Separating real and imaginary parts we get two real alternating series, both of them converge and then the series converge. It does not converge absolutely, $\sum_n \left|\frac{i^n}{\sqrt{n}}\right|$ diverges.