6.2/6.3 Laplace transform, examples and applications

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Laplace transform: definition and existence

A special machine which changes one function into another. Input: $f(t)$, $t > 0$. Output:

$$F(s) = \mathcal{L}\{f\}(s) = \int_{0}^{\infty} f(t)e^{-st} dt$$

Definition

If $f$ is a piece-wise continuous function on each interval $[0, A]$ then

$$F(s) = \mathcal{L}\{f\}(s) = \int_{0}^{\infty} f(t)e^{-st} dt = \lim_{A \to \infty} \int_{0}^{A} f(t)e^{-st} dt,$$

if the limit exists.

Theorem

If $|f(t)| < M e^{kt}$ (such $f$ is called a function of exponential type), then $\mathcal{L}\{f\}(s)$ exists for all $s > k$. 

2
Basic rules

— Linearity
\[ \mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\} \]

— First shift rule
\[ \mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s - a) \]

— Second shift rule
\[ \mathcal{L}\{u_c(t)f(t - c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s), \quad c > 0 \]

— Derivatives
\[ \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0), \quad \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0), \]
\[ \mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \]
Laplace transform of derivatives

Suppose that \( f \) and its derivatives are continuous functions of exponential type. Then integration by parts gives

\[
\mathcal{L}\{f'\} = \int_0^\infty f' e^{-st} \, dt = f(t) e^{-st} \bigg|_0^\infty - \int_0^\infty f(t)(-se^{-st}) \, dt = s\mathcal{L}\{f\} - f(0),
\]

Then for the second derivative, we apply the previous formula twice,

\[
\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0),
\]

For higher order derivatives we get

\[
\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).
\]
Laplace transform of integrals

If \( f \) is a piece-wise continuous function of exponential type then

\[
\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s)
\]

The function \( g(t) = \int_0^t f(\tau) d\tau \) is continuous, of exponential type and \( g'(t) = f(t) \).
Applications to ODE

Solve the initial value problem

\[ ay'' + by' + cy = g, \quad y(0) = K_0, \quad y'(0) = K_1 \]

Apply the Laplace transform

\[(as^2 + bs + c)Y - (as + b)K_0 - aK_1 = G\]

Then

\[ Y(s) = \frac{G(s)}{as^2 + bs + c} + \frac{(as + b)K_0 + aK_1}{as^2 + bs + c} \]

We can find \( \mathcal{L}\{(as^2 + bs + c)^{-1}\} \) and use the inverse Laplace transform to compute \( y \).
Example 1

\[ y'' - 9y = 1, \ y(0) = 1, \ y'(0) = 0 \]

— Apply the Laplace transform

\[ s^2 Y - s - 9Y = \frac{1}{s} \]

— Solve for \( Y \) and use partial fractions to write down the answer

\[ Y(s) = \frac{s^2 + 1}{s(s^2 - 9)} = -\frac{1}{9s} + \frac{0.5}{9(s - 3)} + \frac{0.5}{9(s + 3)} \]

— Find \( y \) by performing the inverse transform

\[ y(t) = -\frac{1}{9} + \frac{1}{18}e^{3t} + \frac{1}{18}e^{-3t} \]
Example 2

\[ y'' + y' - 2y = \sin t, \quad y(0) = 0, \quad y'(0) = 1 \]

1.

\[ s^2 Y - 1 + sY - 2Y = \frac{1}{s^2 + 1} \]

2.

\[ Y(s) = \frac{s^2 + 2}{(s^2 + 1)(s^2 + s - 2)} \]

We want to decompose it using partial fractions (see next slide):

\[ Y(s) = -0.1 \frac{s}{s^2 + 1} - 0.3 \frac{1}{s^2 + 1} + 0.5 \frac{1}{s - 1} - 0.4 \frac{1}{s + 2} \]

3. Applying the inverse transform we get

\[ y(t) = -0.1 \cos t - 0.3 \sin t + 0.5e^t - 0.4e^{-2t} \]
Partial fractions: example

We look for the representation

\[ Y(s) = \frac{as + b}{s^2 + 1} + \frac{c}{s - 1} + \frac{d}{s + 2}, \]

where

\[ s^2 + 2 = (as + b)(s - 1)(s + 2) + c(s^2 + 1)(s + 2) + d(s^2 + 1)(s - 1). \]

Now we either open the brackets, compare the coefficients and solve a system of linear equations or we substitute \( s = 1, 2, i \) and find the constants \( a = -0.1, b = -0.3, c = 0.5, d = -0.4. \)
Laplace transform of discontinuous functions

The building block for discontinuous functions is the step function (Heaviside’s function) $u_c$:

$$u_c(t) = \begin{cases} 
0, & t < c \\
1, & t \geq c 
\end{cases}$$

For $c \geq 0$ we compute its Laplace transform:

$$\mathcal{L}\{u_c\}(s) = \int_c^\infty e^{-st} \, dt = \frac{e^{-cs}}{s}, \quad s > 0.$$  

Further, the change of variables gives the second shift rule

$$\mathcal{L}\{u_c(t)f(t - c)\}(s) = e^{-cs} \mathcal{L}\{f\}(s), \quad c \geq 0.$$  

We use it to evaluate the Laplace transform of piecewise defined functions.
Example

Find the Laplace transform of the function

\[
f(t) = \begin{cases} 
0, & t < 1 \\
(t - 2), & 1 \leq t \leq 3 \\
0, & t > 3
\end{cases}
\]

First we look at the graph of this function

And rewrite the function as

\[
f(t) = (t - 2)(u_3(t) - u_1(t))
= (t - 3)u_3(t) + u_3(t) - (t - 1)u_1(t) + u_1(t)
\]

Then, using the rules above, we compute

\[
\mathcal{L}f(s) = \frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}
\]
Inverse Laplace transform: Example

An important step in the application of the Laplace transform to ODE is to find the inverse Laplace transform of the given function. Find \( f(t) \) such that \( \mathcal{L}\{f\} = F \) is

\[
F(s) = \frac{e^{-2s}}{s^2 + 2s - 3}
\]

First, using the partial functions

\[
\frac{1}{s^2 + 2s - 3} = \frac{1}{4} \left( \frac{1}{s-1} - \frac{1}{s+3} \right).
\]

Then we write

\[
F(s) = \frac{1}{4} \left( \frac{e^{-2s}}{s-1} - \frac{e^{-2s}}{s+3} \right)
\]

and using the second shift rule and the table to get

\[
\mathcal{L}^{-1}(F)(t) = \frac{u_2(t)}{4}(e^{t-2} - e^{-3(t-2)})
\]
Consider an RLC circuit consisting of a resistor $R$, inductor $L$, and capacitor $C$, which is driven by a voltage source $v$. Let $q$ be the charge on the capacitor and let the current in the circuit be $i$. By Kirchhoff’s voltage laws

$$L i'(t) + R i(t) + \frac{1}{C} q(t) = v(t)$$
RLC-circuit cont.

\[ Li'(t) + Ri(t) + \frac{1}{C}q(t) = v(t) \]

In this equation: resistance, inductance, capacitance and voltage are known quantities but current and charge are unknown quantities, \( q(t) = \int_0^t i(\tau) d\tau \).
$Li'(t) + Ri(t) + \frac{1}{C}q(t) = v(t)$

In this equation: resistance, inductance, capacitance and voltage are known quantities but current and charge are unknown quantities, $q(t) = \int_0^t i(\tau)d\tau$.

We apply the Laplace transform

$L(sI(s) - i(0)) + RI(s) + \frac{1}{sC}I(s) = V(s)$
RLC-circuit example

Find $i(t)$ in the circuit with

$$R = 50.2\Omega, \ L = 1\ H, \ C = 0.1\ F, \ v(t) = 99.6(u(t) - u(t - 3)), \ i(0) = 0$$
RLC-circuit example

Find $i(t)$ in the circuit with

$$R = 50.2\,\Omega, \quad L = 1\,H, \quad C = 0.1\,F, \quad v(t) = 99.6(u(t) – u(t – 3)), \quad i(0) = 0$$

After the Laplace transform we get

$$(s + 50.2 + 10/s)l(s) = 100(1 – e^{-3s})/s$$

$$l(s) = \frac{99.6(1 – e^{-3s})}{s^2 + 50.2s + 10} = \frac{99.6(1 – e^{-3s})}{49.8} \left( \frac{1}{s + 0.2} – \frac{1}{s + 50} \right)$$

Then

$$i(t) = 2(e^{-0.2t} – e^{-50t} – e^{-0.2(t-3)}u(t – 3) + e^{-50(t-3)}u(t – 3))$$