14.2 Cauchy's integral theorem
14.3 Cauchy's integral formula
14.4 Derivatives of analytic functions

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Properties of the integral

— Linearity

$$\int_C (c_1 f_1(z) + c_2 f_2(z))dz = c_1 \int_C f_1(z)dz + c_2 \int_C f_2(z)dz$$

— If $\tilde{C}$ is the same curve as C with the reverse orientation (and end points z_e and z_0) then $\int_{\tilde{C}} f(z)dz = -\int_C f(z)dz$.

— If C consists of two pieces C_1 and C_2 then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

An important inequality

Theorem

Suppose that f is a continuous function on a curve C. If $|f(z)| \leq M$ for each z on C and L is the length of C then

$$\left| \int_C f(z) dz \right| \leq LM$$

Using a parametrization, we have

$$\left| \int_a^b f(g(t))g'(t)dt \right| \leq \int_a^b |f(g(t))||g'(t)|dt \leq M \int_a^b |g'(t)|dt = ML$$

Simply connected domains and Cauchy's integral theorem

A domain D on the complex plain is said to be simply connected if any simple closed curve in D is a boundary of a subdomain of D.

Example

1. Any circle is a simply connected domain.
2. A circular ring or a punched disc are not simply connected domains.

Theorem

Let f be an analytic function in a simply connected domain D. If C is a simple closed curve in D then

$$\oint_C f(z)dz = 0$$

Integration of analytic functions along paths

Corollary

In a simply connected domain the integral $\int_C f(z)dz$ of an analytic function does not depend on the path C but only on its end points, we write also

$$\int_C f(z)dz = \int_{z_0}^{z_e} f(z)dz$$

Anti-derivative

Theorem

Let f be an analytic function on a simply connected domain D. Then there is an analytic function F in D such that $F'(z) = f(z)$ for each z in D and

$$\int_C f(z)dz = F(z_e) - F(z_0)$$

where C is a simple curve with end points z_0 and z_e.

To construct the anti-derivative we fix some point z_c in D and for each z in D define

$$F(z) = \int_{z_c}^{z} f(\zeta)d\zeta$$

One can check that F defined in this way is analytic and $F'(z) = f(z)$.

Cauchy's integral formula

Theorem

Let f be an analytic function in a simply connected domain D. If C is a simple closed curve in D that encloses a point z_0 then

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

To prove the formula we write $f(z) = f(z_0) + (f(z) - f(z_0))$ and

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + \frac{f(z)-f(z_0)}{z-z_0}$$

The proof for Cauchy's integral theorem implies that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_K \frac{f(z)}{z-z_0}\,dz$$

where K is a small circle centered at z_0.

Proof of Cauchy’sintegral formula

After replacing the integral over C with one over K we obtain

$$f(z_0)\int_K \frac{1}{z-z_0}dz + \int_K \frac{f(z)-f(z_0)}{z-z_0}dz$$

The first integral is equal to $2\pi i$ and does not depend on the radius of the circle. The second one converges to zero when the radius goes to zero (by the ML-inequality).

Example

Assume that C encloses z_0 then

- $\oint_C \frac{z}{z-z_0} dz = 2\pi i z_0$
- $\oint_C \frac{e^z}{z-z_0} dz = 2\pi i e^{z_0}$
- $\oint_C \frac{z^3}{z^2+1} dz = \oint_C \frac{z^3}{(z+i)(z-i)} dz = \frac{1}{2i} \left(\oint_C \frac{z^3}{z-i} dz - \oint_C \frac{z^3}{z+i} dz \right)$
  If C encloses both i and $-i$ then we apply the Cauchy's formula to both integrals

$$\oint_C \frac{z^3}{z^2+1} dz = \frac{1}{2i}(2\pi i i^3 - 2\pi i(-i)^3) = -2\pi i$$

Derivatives of analytic functions

Theorem

If f is analytic in some domain D then it has derivatives of any order which are also analytic functions. Moreover,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz,$$

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz,$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a simple closed path in D that bounds some domain in D which contains z_0.

Examples

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$$\oint_C \frac{z}{(z-z_0)^2}dz = 2\pi i$$

—

$$\oint_C \frac{e^z}{(z-z_0)^4}dz = \frac{2\pi i e^{z_0}}{6}, \quad (e^z)''' = e^z$$

—

$$\oint_C \frac{\cos z}{(z-\pi)^3}dz = \frac{-2\pi i \cos\pi}{24} = \frac{\pi i}{12}, \quad (\cos z)'' = -\cos z$$

Cauchy's inequality

Suppose that f is an analytic function in a disc of radius r around z_0 and that $|f(z)| \leq M$ when $|z - z_0| = r$. Then

$$|f(z)| \leq M, \quad \text{and} \quad |f^{(n)}(z_0)| \leq \frac{n!M}{r^n}, \quad \text{for } |z - z_0| \leq r.$$

Let $C = \{z : |z - z_0| = r\}$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

then taking the absolute values and applying ML-inequality, we get

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

Applications

Theorem (Liouville)

If a function is analytic in a whole complex plane and bounded in absolute value, then it is a constant

Functions analytic in the whole plane are called entire functions.

Theorem (Morera)

If f is continuous in a simply connected domain D and

$$\oint_C f(z)dz = 0$$

for any closed curve C. Then f is analytic in D.