14.2 Cauchy's integral theorem
14.3 Cauchy's integral formula

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Properties of the integral

- Linearity

$$\int_C (c_1 f_1(z) + c_2 f_2(z))dz = c_1 \int_C f_1(z)dz + c_2 \int_C f_2(z)dz$$

- If $\tilde{C}$ is the same curve as C with the reverse orientation (and end points z_e and z_0) then $\int_{\tilde{C}} f(z)dz = -\int_C f(z)dz$.
- If C consists of two pieces C_1 and C_2 then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

An important inequality

Theorem

Suppose that f is a continuous function on a curve C. If $|f(z)| \leq M$ for each z on C and L is the length of C then

$$\left|\int_C f(z)dz\right| \leq LM$$

Using a parametrization, we have

$$\left|\int_a^b f(g(t))g'(t)dt\right| \leq \int_a^b |f(g(t))||g'(t)|dt \leq M\int_a^b |g'(t)|dt = ML$$

Simply connected domains and Cauchy's integral theorem

A domain D on the complex plain is said to be simply connected if any simple closed curve in D is a boundary of a subdomain of D.

Example

1. Any circle is a simply connected domain.
2. A circular ring or a punched disc are not simply connected domains.

Theorem

Let f be an analytic function in a simply connected domain D. If C is a simple closed curve in D then

$$\oint_C f(z)dz = 0$$

Integration of analytic functions along paths

Corollary

In a simply connected domain the integral $\int_C f(z)dz$ of an analytic function does not depend on the path C but only on its end points, we write also

$$\int_C f(z)dz = \int_{z_0}^{z_e} f(z)dz$$

Anti-derivative

Theorem

Let f be an analytic function on a simply connected domain D. Then there is an analytic function F in D such that $F'(z) = f(z)$ for each z in D and

$$\int_C f(z)dz = F(z_e) - F(z_0)$$

where C is a simple curve with end points z_0 and z_e.

To construct the anti-derivative we fix some point z_c in D and for each z in D define

$$F(z) = \int_{z_c}^{z} f(\zeta)d\zeta$$

One can check that F defined in this way is analytic and $F'(z) = f(z)$.

Cauchy's integral formula

Theorem

Let f be an analytic function in a simply connected domain D. If C is a simple closed curve in D that encloses a point z_0 then

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

To prove the formula we write $f(z) = f(z_0) + (f(z) - f(z_0))$ and

$$\frac{f(z)}{z - z_0} = \frac{f(z_0)}{z - z_0} + \frac{f(z) - f(z_0)}{z - z_0}$$

The proof for Cauchy's integral theorem implies that

$$\oint_C \frac{f(z)}{z - z_0} dz = \oint_K \frac{f(z)}{z - z_0} dz$$

where K is a small circle centered at z_0.

Proof of Cauchy's integral formula

After replacing the integral over C with one over K we obtain

$$f(z_0)\int_K \frac{1}{z-z_0}dz + \int_K \frac{f(z)-f(z_0)}{z-z_0}dz$$

The first integral is equal to $2\pi i$ and does not depend on the radius of the circle. The second one converges to zero when the radius goes to zero (by the ML-inequality).

Examples

Assume that C encloses z_0 then

- $\oint_C \frac{z}{z-z_0} dz = 2\pi i z_0$
- $\oint_C \frac{e^z}{z-z_0} dz = 2\pi i e^{z_0}$
- $\oint_C \frac{z^3}{z^2+1} dz = \oint_C \frac{z^3}{(z+i)(z-i)} dz = \frac{1}{2i}\left(\oint_C \frac{z^3}{z-i} dz - \oint_C \frac{z^3}{z+i} dz\right)$
  If C encloses both i and $-i$ then we apply the Cauchy's formula to both integrals

$$\oint_C \frac{z^3}{z^2+1} dz = \frac{1}{2i}(2\pi i i^3 - 2\pi i(-i)^3) = -2\pi i$$