14.1 Line integral in the complex plane

Eugenia Malinnikova, NTNU

October 17, 2017

Definition: Riemann sums

Let C be a smooth simple curve on the complex plain with end points z_0 and z_e. We consider a subdivision of this curve into small pieces by points $z_0, z_1, ..., z_n = z_e$ on the curve and on each part of the curve (z_j, z_{j+1}) we choose an additional point ζ_j. Let f be a continuous complex valued function on C. To a partition $\{z_0, ..., z_n\}$ with choosen points $\{\zeta_0, ...\zeta_{n-1}$ we assign the Riemann sum

$$S_n = f(\zeta_0)(z_1 - z_0) + f(\zeta_1)(z_2 - z_1) + ... + f(\zeta_{n-1})(z_n - z_{n-1})$$

Now,if $n \to \infty$ and the partitions are chosen such that $|z_{j+1} - z_j|$ tend to zero,then the sequence S_n has a limit, by the definition it is $\int_C f(z)dz$.

Parametrization of the curve

Proposition

If a simple curve C is parametrized by a differentiable function $g : [a, b] \to \mathbb{C}$ with $g'(t) \neq 0$. Then

$$\int_C f(z)dz = \int_a^b f(g(t))g'(t)dt$$

A Riemann sum for the integral $\int_a^b f(g(t))g'(t)dt$ is of the form $\sum_k f(g(s_k))g'(s_k)(t_{k+1} - t_k)$, where $a = t_0 < s_0 < t_1 < s_1 < ... < t_n = b$, while a corresponding Riemann sum for the integral of f over C is $\sum_k f(g(s_k))(g(t_{k+1}) - g(t_k))$. The difference between the sums goes to zero as $n \to \infty$.

Example

Let C be a quarter-circle, the part of the unit circle with $z_0 = 1,\ z_e = i$ and let $f(z) = z$. We want to compute $\int_C f(z)dz$.

1. By the definition: take $z_k = \zeta_k = e^{ik\pi/2n}$, $k = 0, ..., n$, then

$$S_n = \sum_{k=0}^{n-1} e^{ik\pi/2n}(e^{i(k+1)\pi/2n} - e^{ik\pi/2n}) = \sum_{k=0}^{n-1} e^{ik\pi/n}(e^{i\pi/2n} - 1)$$

$$= \frac{e^{i\pi} - 1}{e^{i\pi/n} - 1}(e^{i\pi/2n} - 1) = \frac{e^{i\pi} - 1}{e^{i\pi/2n} + 1} \to -1$$

2. By a parametrization, $g(t) = e^{it}, 0 \leq t \leq \pi/2$, $g'(t) = ie^{it}$then

$$\int_C f(z)dz = \int_0^{\pi/2} e^{it} ie^{it} dt = i \int_0^{\pi/2} e^{2it} dt = \frac{1}{2}\ e^{2it}\Big|_{t=0}^{t=\pi/2} = -1$$

An important examples

Let C_R be the circle of radius R centered at the origin, $f(z) = z^m$, where m is integer.

$$I_m(R) = \int_{C_R} z^m dz$$

We parametrize the circle by $g(t) = Re^{it}, 0 \leq t < 2\pi$.

$$I_m(R) = \int_0^{2\pi} R^m e^{itm}(iRe^{it})dt = iR^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt$$

- If $m = 0, 1, 2, 3, ...$.Then $I_m(R) = 0$.
- $f(z) = z^{-1}$, $m = -1$, then $I_{-1}(R) = 2\pi i$.
- $m = -2, -3, ...$ gives $I_m(R) = 0$

The value of $I_m(R)$ does not depend on R.

Shift of variable

Let now $T_R = \{|z - z_0| = R\}$ and $f(z) = (z - z_0)^m$ then one can do a change of variable

$$\int_{T_R} (z - z_0)^m dz = \int_{C_R} z^m dz = \begin{cases} 0, & m \neq -1 \\ 2\pi i, & m = -1 \end{cases}$$

We write $\int_{T_R} f(z)dz$ as $\oint_{T_R} f(z)dz$ to remind that the curve is closed or was $\int_{|z-z_0|=R} f(z)dz$.

Reduction to real valued integrals

Another way to compute the integral of a complex valued function $f(z) = u(z) + iv(z)$ over a curve C is to use that $dz = dx + idy$ and then reduce integration to real integration along curve

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (udy + vdx)$$

Three approaches to integration

- Definition by Riemann sum: provides some intuition and understanding of the integral, can be used to prove main properties;
- Parametrization: is used to compute integrals
- Real valued curve integration: helps to handle analytic functions (see below).

We will study integration of analytic functions.