17.1 Geometry of analytic functions. Conformal mapping

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Geometrical viewpoint

Functions of a complex variable are mappings from $\mathbb{C}$ (or a domain $D \subset \mathbb{C}$ to $\mathbb{C}$, $w = f(z)$, $z \in D$.

D is the *range* of f, $f(D) = \{w = f(z), z \in D\}$is the image of D under action f.
Examples: images of curves, images of domains.
Example from last week: $f(z) = e^z$, images of the Cartesian coordinate net, images of strips, etc.
Our goal for today: get intuition (and some rigorous) knowledge about mapping by analytic functions.

Simplest examples

- Shift: Fix $a \in \mathbb{C}$. Let $w = f(z) = z + a$. This is a shift of the complex plane
- Rotation: Fix $c \in \mathbb{C}$, $|c| = 1$. Let $w = f(z) = cz$. This is a rotation of the complex plane.
- Scaling and rotation Let $b \in \mathbb{C}$ be arbitrary, then $w = f(z) = cz$ is scaling by $|b|$ and rotation by $Arg(b)$.
- Linear mapping: $w = f(z) = a + bz$. This is a combination of shift, scaling and rotation of the complex plane.

Linear mappings preserve the angles between straight lines

In order to have this fact for more general mappings we need a definition of angle between curves.

Angles between curves

Let C_1, C_2 be two curves which intersect at some point z_0.

The angle between C_1 and C_2 at z_0 is the angle between the tangent lines to C_1 and C_2 at this point

A curve (in complex notation) is defined by:

$$C = \{z(t) = x(t) + iy(t), t \in (a, b) \subset \mathbb{R}\}.$$

Examples: arcs, segments in $\mathbb{C}$, arbitrary curves.
Natural parametrization is by the arc length, such that $|\dot{z}(t)| = 1$.

Given $t_0 \in (a, b)$ and the corresponding point $z_0 = z(t_0) \in \mathbb{C}$, $\dot{z}(t_0)$ is directed along the tangent at z_0.

Conformal mapping

A mapping is called conformal if it preserves angles between curves (including the direction).

Examples:

– Linear mappings are conformal

– The exponential mapping is conformal (so far we checked this just for horizontal and vertical lines)

– Reflections with respect lines are not conformal

<u>Fact</u> A mapping $w = f(z)$ by an analytic function f is conformal at each point z where $f'(z) \neq 0$.

The inverse statement is also true: a conformal mapping with partial derivatives is an analytic function which derivative is not zero at each point.

Idea of the proof

If f is analytic near z_0, then locally (i.e. when z is close to z_0)

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{\text{this is the main part}} + o(|z - z_0|),$$

The main part is a linear mapping!
Locally each analytic function is a shift and a scaling with rotation (where $f' \neq 0$).

To prove the inverse statement, one can express mathematically the angles between images of straight lines and deduce the Cauchy-Riemann equations.

Critical points

Definition If $f(z)$ is analytic and if $f'(z_0) = 0$then z_0 is called a critical point of f.

The mapping $w = f(z)$ defined by an analytic function is conformal except at critical points.

Q: What happens in the critical points ?

Power function

$$w = f(z) = z^{\alpha}, \alpha > 0.$$

This is an analytic function at $z \neq 0$ due to our definition of power function $z^{\alpha} = e^{\alpha \ln z}$ and $f'(z) = \alpha z^{\alpha-1}$
It also can be written as

$z = re^{i\phi} \Rightarrow w = r^{\alpha} e^{i\alpha\phi}$ - this mapping opens angles (if $\alpha > 1$) or compress angles (if $\alpha < 1$).

Spacial case: $w = f(z) = z^{\alpha}$, $n > 0$, integer. Then $f(z)$ is an analytic function at $z = 0$ as well, $z = 0$ is a critical point. Each angle with vertex at the origin is mapped into an anglewhich is n-times larger.

Further examples of conformal mappings

- Exponential function $f(z) = e^z$.
- Logarithmic function $f(z) = \ln z$

Definition Let f maps a domain $S \subset \mathbb{C}$ *one to one* onto a domain $T \subset \mathbb{C}$, (i.e. $T = \{w = f(z), z \in S\}$ and for each $w \in T$ there is just one $z \in S$ such that $f(z) = w$) we can define the inverse mapping $f^{-1} : T \to S$:

$$z = f^{-1}(w) \text{ if } w = f(z)$$

Principle of inverse mapping: If f is conformal then f^{-1} is conformal as well

Two more examples

Inversion: $w = f(z) = \frac{1}{z}$
Make pictures. Inversion maps the unit disc onto exterior of the unit disc.

Rule In order to trace the image of a domain we have to look at the image of its boundary.

Joukowski mapping: $w = f(z) = z + 1/z$.

- Derivative and the critical points
- Exterior (and interior) of the unit disc onto exterior of the segment
- Bigger discs onto ellipses
- Shifted discs onto ”Joukowski airfoil”