13.6-7 Trigonometric and hyperbolic functions. Logarithm, power function.

Yurii Lyubarskii, NTNU

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Trigonometric and hyperbolic functions

Using the complex exponential function
$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$, we now can extend the definition of trigonometric and hyperbolic functions to the complex plane:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cos iz, \quad \sinh z = \frac{e^z - e^{-z}}{2} = -i \sin iz$$

Check list for trigonometric and hyperbolic functions

— definition

— derivatives

— zeroes

— expressions for the absolute values

— periodicity

— trigonometric formulas

— solution of equations

— from trigonometrical functions to hyperbolic functions

Logarithm

Setting of the problem: given $w \in \mathbb{C}$ find $z \in \mathbb{C}$ such that $e^z = w$.
Definition $z = \ln w$
$w = re^{i\phi} \Rightarrow z = x + iy = \ln r + i\phi$ is A solution.

$\ln r + i(\phi + 2n\pi)$ also is a solution for any integer n
The argument can be chosen in many ways $\Rightarrow \ln w$ is a multivalued function.
Principal value of the argument: $\mathrm{Arg} w \in (-\pi, \pi]$

Principal value of the logarithm: $\mathrm{Ln} w = \ln |w| + i \mathrm{Arg} w$

Logarithm

Examples.

Logarithm of the product and of the ratio
Derivative: $(\ln z)' = 1/z$. I give a proof as if I already know that $\ln z$ is an analytic function. Check C-R conditions.
Digression: C-R condition in polar form

General power

- definition: $z^a = e^{a \ln z}$
- examples 2^2, 2^i, 2^{1+i}
- examples $z^{1/n}$, $1^{1/n}$, $1^{\sqrt{2}}$.