12.6 Heat equation, 12.2-3 Wave equation

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Heat equation in higher dimensions

The heat equation in higher dimensions (two or three) is

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

We study steady solutions (that does-not depend on time t). Then the equation becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Solutions are called harmonic functions ($\Delta u = 0$),
$\Delta u = u_{xx} + u_{yy} + u_{zz}$, Δ is called the Laplace operator.

Separation of variables for 2D Laplace

We are looking for solutions of $u_{xx} + u_{yy} = 0$ on $0 \le x \le a, 0 \le y \le b$ of the form $u(x, y) = F(x)G(y)$ then the equation becomes

$$\frac{G''}{G} = -\frac{F''}{F} = \mathrm{const}$$

Boundary condition for Laplace equation:

— Dirichlet, function is known on the boundary, $u(0, y), u(a, y), u(x, 0), u(x, b)$ are given;

— Neumann, the normal derivative is known in the boundary $u_x(0, y), u_x((a, y), u_y(x, 0), u_y(x, b)$ are given

— Robin (mixed), function on a part of the boundary and normal derivative on the other part.

Example

We assume that the boundary conditions are:

$$u(0, y) = c_1, \quad u(a, y) = c_2,$$
$$u(x, 0) = u(x, b) = c_1 + (c_2 - c_1)(x/a)^2$$

1. Simplify the boundary condition (for example subtract some simple solution to get $u(0, y) = u(a, y = 0)$)
2. Use separation of variables with the boundary condition in x
3. Combine solutions from separated variables to satisfy the other boundary condition

Simplification

We can look at the function

$$v(x,y) = u(x,y) - c_1 - (c_2 - c_1)(x/a)$$

it is also harmonic (since $k_1 + k_2 x$ is harmonic) and it satisfies

$$v(0,y) = v(a,y) = 0$$

The other boundary condition for v will be

$$v(x,0) = v(x,b) = (c_2 - c_1)((x/a)^2 - x/a) = cx(x-a)$$

Separation of variables

Now we are looking for solutions of the form $F(x)G(y)$ that satisfy $F(0) = F(a) = 0$. We have

$$\frac{G''}{G} = -\frac{F''}{F} = k_n$$

Then $F_n(x) = \sin(n\pi x/a)$, $k_n = -n^2\pi^2/a^2 = -p_n^2$ are solutions and the corresponding G_n are

$$G_n(y) = A_n \cosh p_n y + B_n \sinh p_n y$$

We get $v_n(x, y) = F_n(x) g_n(y)$ are solutions.

Combination of product solutions

Then

$$v(x,y) = \sum_{n=1}^{\infty} v_n(x,y) = \sum_{n=1}^{\infty} (A_n \cosh p_n y + B_n \sinh p_n y) \sin p_n x$$

is also a solution. We want to take a combination which satisfies additional boundary conditions. Boundary conditions on two other sides of the rectangle are

$$v(x,0) = v(x,b) = cx(x-a)$$

We set inn $y = 0$ and $y = b$,

$$cx(x-a) = \sum_{n=1}^{\infty} A_n \sin p_n x = \sum_{n=1}^{\infty} (A_n \cosh p_n b + B_n \sinh p_n b) \sin p_n x$$

We first find A_n from the first equality (take the Fourier series of the function $x(x-a)$ and then find B_n from the second equality:

$$B_n \sinh p_n b = A_n (1 - \cosh p_n b)$$

Fourier series

Let $f(x) = cx(x-a)$ for $0 < x < a$. We extend f to an odd $2a$-periodic function and compute the Fourier coefficients:

$$b_n = \frac{2c}{a}\int_0^a x(x-a)\sin\frac{n\pi x}{a}dx = \begin{cases} -\frac{8ca^3}{n^3\pi^3}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

We take $A_n = b_n$ and $B_n = -b_n \tanh p_n b/2$. Then the solution is

$$v(x,y) = -\frac{8ca^3}{\pi^3}\sum_{k=1}^{\infty}\frac{\sin w_k x}{(2k-1)^3}\left(\cosh w_k y - \tanh\frac{bw_k}{2}\sinh w_k y\right),$$

where $w_k = p_{2k-1} = \frac{(2k-1)\pi}{a}$. Finally, the temperature distribution $u(x,y)$ that we were looking for is given by

$$u(x,y) = c_1 + \frac{(c_2-c_1)x}{a} + v(x,y)$$

Wave equation

We model small vibrations of an elastic homogeneous string, assume that the string performs small motion in vertical direction only.

Physical assumptions:

- The string is homogeneous and elastic.
- The gravitational force can be neglected.
- Each part of the string moves only vertically.

We are looking for a function $u(x, t)$ that describes the motion. The equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}$$

where T is the tension of the string and ρ is the density. The equation is called one-dimensional wave-equation.

Separation of variables

As before, we are looking at solutions of the form $u(x, t) = F(x)G(t)$. And we get coupled equations for F and G:

$$\frac{F''}{F} = \frac{G''}{c^2 G} = k$$

Suppose that two ends of the string are fixed, $u(0, t) = u(L, t) = 0$, then we want to find solutions with $F(0) = F(L) = 0$. As earlier we obtain

$$F_n(x) = \sin \frac{n\pi x}{L}, \ n \text{ is integer}, \quad k_n = -\left(\frac{n\pi}{L}\right)^2 = -p_n^2$$

Then for G we obtain

$$G_n(t) = A_n \cos cp_n t + B_n \sin cp_n t$$

When $n = 1$ we obtain the fundamental mode of the string.

Initial conditions

To describe the motion of the string we need to know its initial position $f(x) = u(x, 0)$ and initial velocity $g(x) = u_t(x, 0)$.
We look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos cp_n t + B_n \sin cp_n t) \sin cp_n x$$

The initial conditions give

$$f(x) = \sum_{n=1}^{\infty} A_n \sin cp_n x, \quad g(x) = \sum_{n=1}^{\infty} cp_n B_n \sin cp_n x$$

Method works for other equations

Example

a) Find all functions of the form $u(x,t) = F(x)G(t)$ that satisfy the differential equation

$$u_t + (1+t^2)(2u - u_{xx}) = 0, \quad 0 \leq x \leq \pi/2,\ t > 0$$

and the boundary conditions $u(0,t) = 0, u_x(\pi/2,t) = 0$.
b) Find the solution of the above equation that also satisfies the initial condition $u(x,0) = \sin 3x + \sin 17x$.

Solution (a), part 1

Separating variables in the equation, we obtain

$$\frac{G'}{(1+t^2)G} + 2 = \frac{F''}{F} = \mathrm{const}$$

The boundary condition implies $F(0) = 0$ and $F'(\pi/2) = 0$. We have from the equation $F'' = cF$ then

- If $c = 0$ we get $F(x) = ax + b$ and boundary conditions give $a = b = 0$ then $F = 0$. It gives only the trivial solution $u = 0$.
- If $c = p^2 > 0$ then $F(x) = ae^{px} + be^{-px}$ (or one can use $F(x) = c_1 \cosh px + c_2 \sinh px$). The boundary conditions give $a + b = 0$ and $pae^{p\pi/2} - pbe^{-p\pi/2} = 0$ then $a = b = 0$, only the trivial solution.
- If $c = -p^2 < 0$ then $F(x) = a\cos px + b\sin px$ and the boundary conditions give $a = 0$, $pb\cos p\pi/2 = 0$. We gat a non-trivial solution when $p = (2k+1)$ with integer k.

$$F_k(x) = \sin(2k+1)x$$

Solution (a), part 2

Then the equation for G is

$$\frac{G_k'}{G_k} = -((2k+1)^2+2)(1+t^2) = -c_k(1+t^2)$$

Integrating, we get

$$\ln G_k(t) = -c_k(t + t^3/3) + C, \quad G_k(t) = C_k e^{-c_k(t+t^3/3)}$$

All solutions of the form $F(x)G(t)$ are given by

$$u_k(x,t) = F_k(x)G_k(t) = C_k \sin(2k+1)x \, e^{-(4k^2+4k+3)(t+t^3/3)}$$

Solution (b)

If we have $u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$ then the initial condition is

$$u(x,0) = \sum_{k=0}^{\infty} C_k \sin(2k+1)x$$

We have $u(x,0) = \sin 3x + \sin 17x$ then $C_1 = C_8 = 1$ and all other coefficients are zeros, we get

$$u(x,t) = \sin 3x \, e^{-11t} + \sin 17x \, e^{-291t}$$