12.1 Basics on PDE, 12.5-6 Heat equation

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Partial derivatives

We will now work with functions of several variables like

$$v(t, x, y, z)$$

and we use a number of standard notation for the partial derivatives:

$$v_x = \partial_x v = \frac{\partial v}{\partial x}$$
$$v_{tt} = \partial_{tt}^2 v = \frac{\partial^2 v}{\partial t^2}$$
$$v_{yz} = \partial_{yz}^2 v = \frac{\partial^2 v}{\partial y \partial z}$$

PDEs

P(artial) D(ifferential) E(quation) is an equation on a function of several variables, which contains its partial derivatives.

Examples $u = u(x,t)$

- $$-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = 0$$
- $$-\frac{\partial u}{\partial t} + (1 + \cos t)\frac{\partial^2 u}{\partial x^2} + u = t$$
- $$-\frac{\partial^2 u}{\partial t^2} + (1 + \cos u)\frac{\partial^3 u}{\partial x^3} + u = 0$$

Simplest classification

The highest derivative involved determines the order of the equation. We distinguish between linear and non-linear equations.

- $$-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = 0$$

  order 2, linear, homogeneous, with constant coefficients.

- $$-\frac{\partial u}{\partial t} + (1 + \cos t)\frac{\partial^2 u}{\partial x^2} + u = t$$

  order 2, linear,non-homogeneous, with non-constant coefficients

- $$-\frac{\partial^2 u}{\partial t^2} + (1 + \cos u)\frac{\partial^3 u}{\partial x^3} + u = 0$$

  order 3, nonlinear

Linear equations

$$Lu = f \quad (*)$$

where L is a linear expression, $L(au + bv) = aL(u) + bL(v)$.
Homogeneous equation $\Leftrightarrow$ $f = 0$

Superposition principle:

if $u_1, u_2, \ldots, u_n$ are solutions to a linear homogeneous equation, then $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ also is a solution to this equation.

Superposition principle (non-homogeneous equation):

If u_1 and u_2 are the solutions of (*), then $u_1 - u_2$ solves the corresponding homogeneous equation $Lu = 0$

PDE are models for real processes:

- $u = u(x,t)$, $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ 1D wave equation;
- $u = u(x,t)$, $\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2}$ 1D heat equation;
- $u = u(x,y)$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 2D Laplace equation
- $u = u(x,y)$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ 2D Poisson equation

List of questions:

- Does the PDE have any solution?
- What kind of additional data should we specify in order to solve PDE and described a given process?
- Is the solution unique and how do we find it?

And this is only the beginning (stability, singularity, etc.)

Heat equation

We consider the temperature in a long thin metal bar of is perfectly insulated laterally and the heat flows only through the ends.

Physical assumptions:

- The heat energy of a body is equal to σmu, where m is the mass,u is temperature and σ is the specific heat.
- The law of heat transfer: The rate of heat transfer is proportional to temperature gradient.
- Conservation of energy

The equation we get is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and in higher dimensions

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Separation of variables

Assume that there is a solution of the 1D heat equation of the form $u(x,t) = F(x)G(t)$ then the equation becomes

$$\frac{G'}{c^2 G} = \frac{F''}{F}$$

Assume in addition that we have boundary condition $u(0,t) = u(L,t) = 0$ (ends of the bar are kept on zero temperature) or $u_x(0,t) = u_x(L,t) = 0$ (the ends are insulated) then we can show that if u is not the zero solution,then the last quantity is negative, so we solve

$$\frac{G'}{c^2 G} = \frac{F''}{F} = -p^2$$

Boundary condition

The last equation gives $F(x) = A\cos px + B\sin px$, if we also have $F(0) = F(L) = 0$. Then

$$F_n(x) = \sin\frac{n\pi x}{L}$$

and $p_n = n\pi/L$. Further,

$$G_n(t) = B_n e^{-c^2 p_n^2 t}$$

We get a family of solutions, $u_n(x,t) = F_n(x)G_n(t)$.

Initial condition: Fourier series

To specify one solution we should use given initial condition$u(x, 0) = f(x)$ (the temperature at the zero moment is given).We assume that we can find $u(x, t) = \sum_n u_n(x, t)$ then

$$u(x, 0) = \sum_n u_n(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

we can find B_n from the sine Fourier series of f (defined on $[0, L]$ and extended to an odd $2L$-periodic function. We get

$$u(x, t) = \sum_n u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-c^2 p_n^2 t}$$