12.1 Basics on PDE, 12.5-6 Heat equation

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September 25, 2017
Partial derivatives

We will now work with functions of several variables like

\[ v(t, x, y, z) \]

and we use a number of standard notation for the partial derivatives:

\[ v_x = \frac{\partial v}{\partial x} \]
\[ v_{tt} = \frac{\partial^2 v}{\partial t^2} \]
\[ v_{yz} = \frac{\partial^2 v}{\partial y \partial z} \]
PDEs

P(artial) D(ifferential) E(quation) is an equation on a function of several variables, which contains its partial derivatives.

Examples $u = u(x, t)$

$-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = 0$

$-\frac{\partial u}{\partial t} + (1 + \cos t)\frac{\partial^2 u}{\partial x^2} + u = t$

$-\frac{\partial^2 u}{\partial t^2} + (1 + \cos u)\frac{\partial^3 u}{\partial x^3} + u = 0$
Simplest classification

The highest derivative involved determines the order of the equation. We distinguish between linear and non-linear equations.

\[-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = 0\]

order 2, linear, homogeneous, with constant coefficients.

\[-\frac{\partial u}{\partial t} + (1 + \cos t)\frac{\partial^2 u}{\partial x^2} + u = t\]

order 2, linear, non-homogeneous, with non-constant coefficients

\[-\frac{\partial^2 u}{\partial t^2} + (1 + \cos u)\frac{\partial^3 u}{\partial x^3} + u = 0\]

order 3, nonlinear
Linear equations

\[ Lu = f \quad (*) \]

where \( L \) is a linear expression, \( L(au + bv) = aL(u) + bL(v) \).

Homogeneous equation \( \iff f = 0 \)

Superposition principle:

if \( u_1, u_2, \ldots, u_n \) are solutions to a linear homogeneous equation, then \( c_1u_1 + c_2u_2 + \ldots + c_nu_n \) also is a solution to this equation.

Superposition principle (non-homogeneous equation):
If \( u_1 \) and \( u_2 \) are the solutions of \( (*) \), then \( u_1 - u_2 \) solves the corresponding homogeneous equation \( Lu = 0 \)
PDE are models for real processes:

— $u = u(x, t), \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ 1D wave equation;
— $u = u(x, t), \quad \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ 1D heat equation;
— $u = u(x, y), \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 2D Laplace equation
— $u = u(x, y), \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ 2D Poisson equation

List of questions:

— Does the PDE have any solution?
— What kind of additional data should we specify in order to solve PDE and described a given process?
— Is the solution unique and how do we find it?

And this is only the beginning (stability, singularity, etc.)
Heat equation

We consider the temperature in a long thin metal bar of is perfectly insulated laterally and the heat flows only through the ends.

Physical assumptions:

— The heat energy of a body is equal to $\sigma m u$, where $m$ is the mass, $u$ is temperature and $\sigma$ is the specific heat.

— The law of heat transfer: The rate of heat transfer is proportional to temperature gradient.

— Conservation of energy

The equation we get is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and in higher dimensions

$$\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$
Separation of variables

Assume that there is a solution of the 1D heat equation of the form $u(x, t) = F(x)G(t)$ then the equation becomes

$$\frac{G'}{c^2 G} = \frac{F''}{F}$$

Assume in addition that we have boundary condition $u(0, t) = u(L, t) = 0$ (ends of the bar are kept on zero temperature) or $u_x(0, t) = u_x(L, t) = 0$ (the ends are insulated) then we can show that if $u$ is not the zero solution, then the last quantity is negative, so we solve

$$\frac{G'}{c^2 G} = \frac{F''}{F} = -p^2$$
Boundary condition

The last equation gives $F(x) = A \cos px + B \sin px$, if we also have $F(0) = F(L) = 0$. Then

$$F_n(x) = \sin \frac{n\pi x}{L}$$

and $p_n = n\pi / L$. Further,

$$G_n(t) = B_n e^{-c^2 p_n^2 t}$$

We get a family of solutions, $u_n(x, t) = F_n(x)G_n(t)$. 
Initial condition: Fourier series

To specify one solution we should use given initial condition \( u(x, 0) = f(x) \) (the temperature at the zero moment is given). We assume that we can find \( u(x, t) = \sum_n u_n(x, t) \) then

\[
u(x, 0) = \sum_n u_n(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}
\]

we can find \( B_n \) from the sine Fourier series of \( f \) (defined on \([0, L]\) and extended to an odd \(2L\)-periodic function. We get

\[
u(x, t) = \sum_n u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-c^2 p_n^2 t}
\]