11.7-11.9 Fourier Transform

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Fourier Transform

Let f be a piece-wise continuous function with finite integral $\int_{-\infty}^{\infty} |f(x)|dx$, we define the Fourier transform of f by

$$\mathcal{F}(f)(w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixw}dx$$

It is a continuous function.

Inverse Fourier Transform

From our experience with the Fourier series expansion we expect that

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw.$$

This is called the inversion formula. It allows to reconstruct the function from its Fourier transform and gives a representation of a function as a combination of the exponential ones.
The formula holds for example when f is piece wise continuous and has left and right derivatives at each point, then the identity holds at points x where f is continuous, otherwise we get $(f(x+0) + f(x-0))/2$
If $\hat{f}$ has finite integral $\int_{-\infty}^{\infty} |\hat{f}(w)| dw < \infty$ then at each point$x$ where f is continuous the inversion formula holds.

Basic properties

- The Fourier transform is linear $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$.
- The Fourier transform of the derivatives
  $\mathcal{F}(f')(w) = iw\mathcal{F}(f)(w) \quad \mathcal{F}(f'')(w) = -w^2\mathcal{F}(f)(w)$
- Time shift $\mathcal{F}(f(x-a))(w) = e^{-iaw}\mathcal{F}(f)(w)$
- Frequency shift $\mathcal{F}(f)(w-b) = \mathcal{F}(f(x)e^{ibx})(w)$
- Convolution $\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)$,

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

Example 1

Example

Let $f(x) = 1, \ -1 < x < 1$ and $f(x) = 0$ otherwise (see also the last lecture). Then

$$\hat{f}(w) = \frac{i}{\sqrt{2\pi} w} \left(e^{-iw} - e^{iw} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

Then the inversion formula gives

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin w}{w} e^{ixw} dw$$

Separating the real and imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin w \cos xw}{w} = \begin{cases} \pi, \ -1 < x < 1 \\ \pi/2, \ x = \pm 1 \\ 0, \ |x| > 1 \end{cases}$$

Example 2

Example

Let $f(x) = e^{-|x|}$ then

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|-iwx} dx =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cos xw dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \sin xw dx$$

Since $f(x)$ is an even function, the second integral equals zero and we have

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-|x|} \cos wx dx = \hat{f}_c(w)$$

the last integral is called the cosine Fourier transform (the calculation works for even function f). Note that $\hat{f}_c$ is even by the definition.

Example 2, continued

We compute the Fourier transform

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-(1-iw)x} + e^{-(1+iw)x}}{2} dx = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}$$

The inversion formula then implies

$$e^{-|x|} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx dw = \frac{2}{\pi} \int_0^\infty \frac{\cos wx}{1+w^2} dx$$

Real form of the Fourier transform

For general f we write its Fourier transform $\hat{f}(w)$ as

$$\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} f(x)\cos wx dx - i\int_{-\infty}^{\infty} f(x)\sin wx dx\right) = a(w) - ib(w)$$

where a is even and b is odd. (Notation in the book: $A = \sqrt{2/\pi}a$, $B = \sqrt{2\pi}b$.) Then the inversion formula gives

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(a(w) - ib(w))e^{iwx}dw =$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} a(w)\cos wx dw + \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} b(w)\sin wx dw$$

Fourier transform of even and odd functions

For even function, we have seen that

$$\hat{f}(w) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

and the function is reconstructed from its Fourier Cosine transform as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \cos wx \, dx$$

Similarly, if f is odd, we have

$$\hat{f}(w) = -i\hat{f}_s(w), \quad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

and we use the Sine Fourier transform for reconstruction

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dx$$

One more example

Compute the Fourier transform of $f(x) = e^{-x^2/2}$. Let $g(w) = \mathcal{F}(f)(w)$ then

$$g'(w) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-xe^{-x^2/2}) e^{-iwx} dx =$$

$$= i\mathcal{F}(f')(w) = i(-iw)\mathcal{F}(f)(w) = -wg(w)$$

Then $g'(w) = -wg(w)$ and $g(w) = Ce^{-w^2/2}$. From the inversion formula, we obtain $C^2 = 1$ and it is not difficult to see that $C > 0$ then $C = 1$.
More generally,

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$$

Applications of the Fourier transform

- Signal analysis, filters
- Differential equations and PDE
- Convolution equations