11.7-11.9 Fourier Transform

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Fourier Transform

Let $f$ be a piece-wise continuous function with finite integral $\int_{-\infty}^{\infty} |f(x)| \, dx$, we define the Fourier transform of $f$ by

$$\mathcal{F}(f)(w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} \, dx$$

It is a continuous function.
Inverse Fourier Transform

From our experience with the Fourier series expansion we expect that

\[ f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{ixw} \, dw. \]

This is called the inversion formula. It allows to reconstruct the function from its Fourier transform and gives a representation of a function as a combination of the exponential ones.

The formula holds for example when \( f \) is piece wise continuous and has left and right derivatives at each point, then the identity holds at points \( x \) where \( f \) is continuous, otherwise we get

\[ (f(x + 0) + f(x - 0))/2 \]

If \( \hat{f} \) has finite integral \( \int_{-\infty}^{\infty} |\hat{f}(w)| \, dw < \infty \) then at each point \( x \) where \( f \) is continuous the inversion formula holds.
Basic properties

— The Fourier transform is linear $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$.

— The Fourier transform of the derivatives

$\mathcal{F}(f'(x))(w) = iw\mathcal{F}(f(x))(w)$

$\mathcal{F}(f''(x))(w) = -w^2\mathcal{F}(f(x))(w)$

— Time shift $\mathcal{F}(f(x - a))(w) = e^{-iaw}\mathcal{F}(f(x))(w)$

— Frequency shift $\mathcal{F}(f(x))(w - b) = \mathcal{F}(f(x)e^{ibx})(w)$

— Convolution $\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)$, 

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$
Example 1

Example

Let \( f(x) = 1, \ -1 < x < 1 \) and \( f(x) = 0 \) otherwise (see also the last lecture). Then

\[
\hat{f}(w) = \frac{i}{\sqrt{2\pi w}} \left( e^{-iw} - e^{iw} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}
\]

Then the inversion formula gives

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} \, dw = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin w}{w} e^{ixw} \, dw
\]

Separating the real and imaginary parts, we obtain

\[
\int_{-\infty}^{\infty} \frac{\sin w \cos xw}{w} = \begin{cases} 
\pi, & -1 < x < 1 \\
\pi/2, & x = \pm 1 \\
0, & |x| > 1
\end{cases}
\]
Example 2

Example

Let \( f(x) = e^{-|x|} \) then

\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-iwx} \, dx =
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cos wx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \sin wx \, dx
\]

Since \( f(x) \) is an even function, the second integral equals zero and we have

\[
\hat{f}(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-|x|} \cos wx \, dx = \hat{f}_c(w)
\]

the last integral is called the cosine Fourier transform (the calculation works for even function \( f \)). Note that \( \hat{f}_c \) is even by the definition.
Example 2, continued

We compute the Fourier transform

\[ \hat{f}(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{e^{-(1-iw)x} + e^{-(1+iw)x}}{2} \, dx = \sqrt{\frac{2}{\pi}} \frac{1}{1 + w^2} \]

The inversion formula then implies

\[ e^{-|x|} = f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c(w) \cos wx \, dw = \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos wx}{1 + w^2} \, dx \]
Real form of the Fourier transform

For general $f$ we write its Fourier transform $\hat{f}(w)$ as

$$
\frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} f(x) \cos wx \, dx - i \int_{-\infty}^{\infty} f(x) \sin wx \, dx \right) = a(w) - ib(w)
$$

where $a$ is even and $b$ is odd. (Notation in the book: $A = \sqrt{2/\pi} a, B = \sqrt{2\pi} b$.) Then the inversion formula gives

$$
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (a(w) - ib(w)) e^{iwx} \, dw =
$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} a(w) \cos wx \, dw + \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} b(w) \sin wx \, dw
$$
Fourier transform of even and odd functions

For even function, we have seen that

\[
\hat{f}(w) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx \, dx
\]

and the function is reconstructed from its Fourier Cosine transform as

\[
f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s(w) \cos wx \, dx
\]

Similarly, if \( f \) is odd, we have

\[
\hat{f}(w) = -i\hat{f}_s(w), \quad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin wx \, dx
\]

and we use the Sine Fourier transform for reconstruction

\[
f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s(w) \sin wx \, dx
\]
One more example

Compute the Fourier transform of $f(x) = e^{-x^2/2}$. Let $g(w) = \mathcal{F}(f)(w)$ then

$$g'(w) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-xe^{-x^2/2})e^{-iwx} dx =$$

$$= i\mathcal{F}(f')(w) = i(-iw)\mathcal{F}(f)(w) = -wg(w)$$

Then $g'(w) = -wg(w)$ and $g(w) = Ce^{-w^2/2}$. From the inversion formula, we obtain $C^2 = 1$ and it is not difficult to see that $C > 0$ then $C = 1$.

More generally,

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$$
Applications of the Fourier transform

— Signal analysis, filters
— Differential equations and PDE
— Convolution equations