11.9 Discrete and Fast Fourier Transform (DFT and FFT)
11.7 Fourier Integral

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September 18, 2017
Fourier series: complex form, review

Let $f$ be a $2L$-periodic function, then its Fourier series is defined by

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-\frac{in\pi t}{L}} \, dt$$

If $f$ is continuous (also at the end-points of the period) then

$$S_f(x) = f(x), \quad \text{otherwise} \quad S_f(x) = \frac{1}{2} \left( f(x+) + f(x-) \right).$$
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**Analysis**: Given $f$ compute $\{c_n\}$

**Synthesis**: Given $\{c_n\}$ approximate/determine $S_f$. 
Analysis of real signals

— Mathematical formulas (integrals)
— Mechanical systems response to a signal (resonance effect), built in our ears (see the last lecture)
— Discrete approximation (is done by computers)
Discrete Fourier Transform (DFT)

Assume that we measure the signal $f$ a discrete sequence of points

$$
 f(0), \ f(2L/N), \ ..., \ f(2L(N - 1)/N)
$$

think of $f$ as of a periodic function and approximate the integrals for Fourier coefficients by Riemann sums, we consider

$$
\tilde{c}_j, j = -N/2, -N/2 + 1, ..., N/2 - 1.
$$
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\tilde{c}_{j,N} = \frac{1}{2LN} \sum_{k=0}^{N-1} f \left( \frac{2Lk}{N} \right) e^{-2ikj\pi/N}, \quad j = 0, \pm 1, \pm 2, \ldots
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It is not difficult to see that $\tilde{c}_{j+N,N} = \tilde{c}_{j,N}$. We will look for $\tilde{c}_{-N/2,N}, \ldots, \tilde{c}_{N/2-1,N}$ only (when $N$ is even). But since $\tilde{c}_{j+N,N} = \tilde{c}_{j,N}$ we can compute

$$\tilde{c}_{0,N}, \tilde{c}_{1,N}, \ldots, \tilde{c}_{N-1,N}.$$
DFT Matrix

To compute the vector of coefficients $\tilde{c}_N = [\tilde{c}_0, N, ..., \tilde{c}_{N-1}, N]^T$ we multiply the vector of measurements of $f$

$$f_N = [f(0), f(2L/N), ..., f(2L(N - 1)/N)]^T$$

by the following sequences

$$1, \ w^j, \ w^{2j}, ..., \ w^{(n-1)j}, \ j = 0, 1, ..., N - 1$$

where $w = e^{-2i\pi/N}$. Thus $\tilde{c}_N = \frac{1}{2NL} F_N f_N$, where $F_N$ is DFT-matrix,

\[
F_N = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w & w^2 & \cdots & w^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2}
\end{bmatrix}
\]
Fast Fourier Transform (FFT)

We have $\hat{f}_N = \tilde{c}_N = \frac{1}{2NL} F_N f_N$ and this computation requires $N^2$ multiplications. The Fast Fourier Transform is a family of algorithms which allow to reduce the complexity of computation substantially.
Fast Fourier Transform (FFT)

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The original idea is to compute the DFT for \( N = 2M \) using the symmetry of the matrix \( F_N \). Divide \( f_N \) into the even and odd parts \( f_e = f_M \) and \( f_o = (f(x + 1/N))_M \) and let \( \hat{f}_e \) and \( \hat{f}_o \) be their Discrete Fourier Transforms (of length \( M \)). Then

\[
\hat{f}_{j,N} = \frac{1}{2}((\hat{f}_e)_{j,M} + w^j(\hat{f}_o)_{j,M}), \quad j = 0, \ldots, M - 1
\]

\[
\hat{f}_{j+M,N} = \frac{1}{2}((\hat{f}_e)_{j,M} - w^j(\hat{f}_o)_{j,M}), \quad j = 0, \ldots, M - 1
\]

FFT requires roughly \( N \log_2 N \) operations!
From Fourier series to Fourier Integral: Example

Let \( f(x) = 1 \) when \(-1 < x < 1\) and \( f(x) = 0 \) otherwise. We extend \( f \) to a \( 2L \)-periodic functions and compute its Fourier coefficients. The Fourier series represents \( f \) as an infinite sum of the periodic functions with periods \( 2L, 2L/2, 2L/3, 2L/4, \ldots, 2L/n, \ldots \) and 

\[
c_0 = \frac{1}{L}, \quad \text{while} \quad c_n = \frac{1}{2L} \int_{-1}^{1} e^{-in\pi x/L} \, dx = \frac{i}{2n\pi} (e^{-in\pi/L} - e^{in\pi/L}) = \frac{\sin \frac{n\pi}{L}}{n\pi}
\]

\[w = \frac{n\pi}{L} \quad L = \frac{\pi}{2}\]
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From discrete to continuous frequencies (Analysis)

Let $f$ be a function on the real line that decays at $\pm \infty$ rapidly enough. Then we define

$$F(w) = \int_{-\infty}^{\infty} f(x)e^{-ixw} \, dx$$

$F$ is called the Fourier integral of $f$. 
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$F$ is called the Fourier integral of $f$. We have

$$\frac{1}{\pi} F(w) = \lim_{L \to \infty} \frac{1}{\pi} \int_{-L}^{L} f(x) e^{-iLwx/L} \, dx,$$

$F(w)$ has all information about the Fourier coefficients of $f$. We also want to represent $f$ through $F$ (Synthesis).
From discrete to continuous frequencies (Synthesis)

We have

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/L} = \frac{1}{L} \sum_{n=-\infty}^{\infty} Lc_n e^{i\pi nx/L} \rightarrow \int_{-\infty}^{\infty} F(w) e^{iwx} \, dw
\]

We get a representation of \( f \) as a Fourier integral

\[
f(x) = \int_{-\infty}^{\infty} F(w) e^{iwx} \, dw
\]
Fourier Transform and Inverse Fourier Transform: formal definition

We define

$$\hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} \, dx$$

$\hat{f}$ is called the Fourier transform of $f$. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} \, dw.$$ 

This is called the inversion formula. It allows to reconstruct the function from its Fourier transform and gives a representation of a function as a combination of the exponential ones.