11.9 Discrete and Fast Fourier Transform (DFT and FFT)
11.7 Fourier Integral

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Fourier series:complex form, review

Let f be a $2L$-periodic function, then its Fourier series is defined by

$$S_f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-\frac{in\pi x}{L}} dx$$

If f is continuous (also at the end-points of the period) then

$$S_f(x) = f(x), \quad \text{otherwise} \quad S_f(x) = \frac{1}{2}\left(f(x+) + f(x-)\right).$$

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Analysis : Given f compute $\{c_n\}$
Synthesis: Given $\{c_n\}$ approximate/determine S_f.

Analysis of real signals

— Mathematical formulas (integrals)

— Mechanical systems response to a signal (resonance effect), built in our ears (see the last lecture)

— Discrete approximation (is done by computers)

Discrete Fourier Transform (DFT)

Assume that we measure the signal f a discrete sequence of points

$$f(0),\ f(2L/N),\ ,...,\ f(2L(N-1)/N)$$

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$$\tilde{c}_{j,N} = \frac{1}{2LN} \sum_{k=0}^{N-1} f\left(\frac{2Lk}{N}\right) e^{-2ikj\pi/N}, \quad j = 0, \pm 1, \pm 2, ...$$

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It is not difficult to see that $\tilde{c}_{j+N,N} = \tilde{c}_{j,N}$. We will look for $\tilde{c}_{-N/2,N}, ..., \tilde{c}_{N/2-1,N}$ only (when N is even). But since $\tilde{c}_{j+N,N} = \tilde{c}_{j,N}$ we can compute

$$\tilde{c}_{0,N}, \tilde{c}_{1,N}, ..., \tilde{c}_{N-1,N}.$$

DFT Matrix

To compute the vector of coefficients $\tilde{c}_N = [\tilde{c}_{0,N}, ..., \tilde{c}_{N-1,N}]^T$ we multiply the vector of measurements of f

$$f_N = [f(0),\ f(2L/N),\ ,...,\ f(2L(N-1)/N)]^T$$

by the following sequences

$$1,\ w^j,\ w^{2j}, ...,\ w^{(n-1)j}, \quad j = 0, 1, ..., N-1$$

where $w = e^{-2i\pi/N}$. Thus $\tilde{c}_N = \frac{1}{2NL} F_N f_N$, where F_N is DFT-matrix,

$$F_N = \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & w & w^2 & ... & w^{N-1} \\ ... & ... & ... & \ddots & ... \\ 1 & w^{N-1} & w^{2(N-1)} & ... & w^{(N-1)^2} \end{bmatrix}$$

Fast Fourier Transform (FFT)

We have $\hat{f}_N = \tilde{c}_N = \frac{1}{2NL} F_N f_N$ and this computation requires N^2 multiplications.The Fast Fourier Transform is a family of algorithm which allow to reduce the complexity of computation substantially.

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The original idea is to compute the DFT for $N = 2M$ using the symmetry of the matrix F_N. Divide f_N into the even and odd parts $f_e = f_M$ and $f_o = (f(x + 1/N))_M$ and let $\hat{f}_e$ and $\hat{f}_o$ be their Discrete Fourier Transforms (of length M). Then

$$\hat{f}_{j,N} = \frac{1}{2}((\hat{f}_e)_{j,M} + w^j(\hat{f}_o)_{j,M}), \quad j = 0, ..., M-1$$

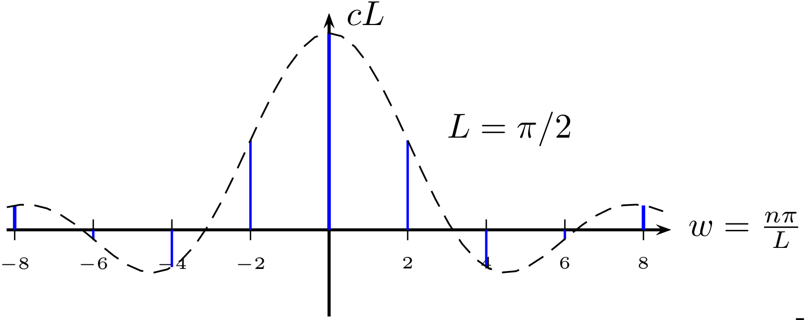
$$\hat{f}_{j+M,N} = \frac{1}{2}((\hat{f}_e)_{j,M} - w^j(\hat{f}_o)_{j,M}), \quad j = 0, ..., M-1$$

FFT requires roughly $N \log_2 N$ operations!

From Fourier series to Fourier Integral: Example

Let $f(x) = 1$ when $-1 < x < 1$ and $f(x) = 0$ otherwise. We extend f to a $2L$-periodic functions and compute its Fourier coefficients. The Fourier series represents f as an infinite sum of the periodic functions with periods $2L$, $2L/2$, $2L/3$, $2L/4$,..., $2L/n$,... and $c_0 = 1/L$, while

$$c_n = \frac{1}{2L}\int_{-1}^{1} e^{-in\pi x/L}dx = \frac{i}{2n\pi}(e^{-in\pi/L} - e^{in\pi/L}) = \frac{\sin\frac{n\pi}{L}}{n\pi}$$



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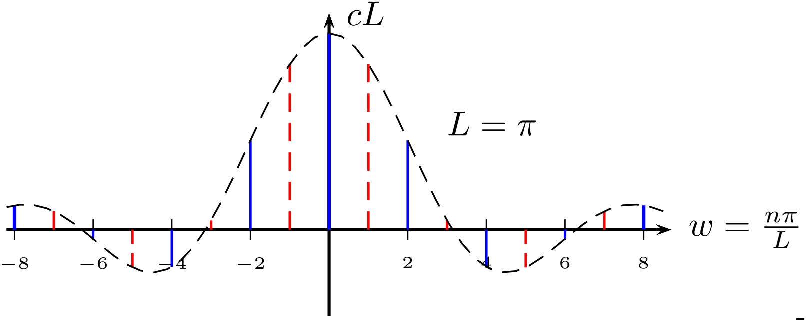

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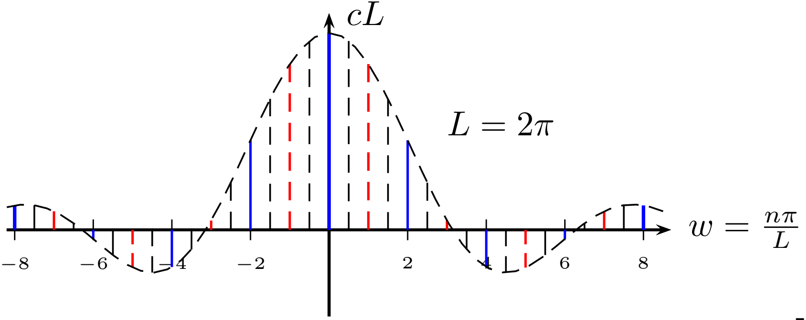

From discrete to continuous frequencies (Analysis)

Let f be a function on the real line that decays at $\pm\infty$ rapidly enough. Then we define

$$F(w) = \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$$

F is called the Fourier integral of f.

From discrete to continuous frequencies (Analysis)

Let f be a function on the real line that decays at $\pm\infty$ rapidly enough. Then we define

$$F(w) = \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$$

F is called the Fourier integral of f. We have

$$\frac{1}{\pi} F(w) = \lim_{L \to \infty} \frac{1}{\pi} \int_{-L}^{L} f(x) e^{-iLwx/L} dx,$$

$F(w)$ has all information about the Fourier coefficients of f. We also want to represent f through F (Synthesis).

From discrete to continuous frequencies (Synthesis)

We have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/L} = \frac{1}{L} \sum_{n=-\infty}^{\infty} L c_n e^{i\pi nx/L} \to \int_{-\infty}^{\infty} F(w) e^{iwx} dw$$

We get a representation of f as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} F(w) e^{iwx} dw$$

Fourier Transform and Inverse Fourier Transform: formal definition

We define

$$\hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$$

$\hat{f}$ is called the Fourier transform of f. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw.$$

This is called the inversion formula. It allows to reconstruct the function from its Fourier transform and gives a representation of a function as a combination of the exponential ones.