11.3. Fourier series and forced oscillation
11.4 Approximation by trigonometric polynomials

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Forced oscillation, Fourier series

We consider linear second order differential equations with constant coefficients:

$$ay'' + by' + cy = f(x)$$

The idea is to expand f into a Fourier series and solve equations

$$\begin{aligned} ay'' + by' + cy &= a_0 \\ ay'' + by' + cy &= a_n \cos nx + b_n \sin nx \end{aligned}$$

using the method of undetermined coefficients.

Example

Let $f(x) = |x|, \ -\pi \leq x \leq \pi$ and f is extended to a 2π-periodic function. The Fourier series of f is

$$S_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

We want to solve the equation $y'' + \omega^2 y = f(x)$. Let solve it for each term of the form $a_n \cos nx$ by the method of undetermined coefficients. The particular solution is

$$y_n(x) = \frac{a_n}{\omega^2 - n^2} \cos nx$$

Then the solution is given by the series

$$y(x) = \frac{\pi}{2\omega^2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2(\omega^2 - (2k-1)^2)}$$

Separating frequencies

The formula from the last example gives:
If $f(x) = a_0 + \sum a_n \cos nx + b_n \sin nx$ and we solve the equation $y'' + \omega^2 y = 0$ then

$$y(x) = \frac{a_0}{\omega_0^2} + \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{\omega^2 - n^2}$$

In this sum the nth coefficients are divided by $\omega^2 - n^2$. When $\omega \approx n_0$ we get $y(x)$ with the coefficient on place n_0 multiplied by a large number and other coefficients comparable or smaller then those of f.

This method of Fourier analysis of signals is built in our ears (cochlea).

Approximation by trigonometric polynomials

Let f be a function on $[-\pi, \pi]$. We are looking for the best approximation of f by a trigonometric polynomial of degree n, i.e. function of the form

$$T_n(x) = A_0 + \sum_{k=1}^{n} (A_k \cos kx + B_k \sin kx),$$

where the error is calculated as

$$E^2 = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^2 dx.$$

Best approximation

Theorem

Let $S_f = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ be the Fourier series of piece-wise continuous function f. Then the best approximation by a trigonometric polynomial of degree n is achieved with the polynomial

$$P_n(x) = a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

In this case the error is given by

$$E_n^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi a_0^2 - \pi \sum_{k=1}^{n}(a_k^2 + b_k^2)$$

Bessel's inequality and Parseval's identity

Since $E_n \geq 0$ the theorem above implies that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \geq 2\pi a_0^2 + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2)$$

It turns out that as n goes to infinity the error tends to zero and the following identity holds

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = 2\pi \sum_{k=-\infty}^{\infty} .|c_k|^2.$$

The Parseval's identity gives also the following formula for the error

$$E_n^2 = \pi \sum_{k=n+1}^{\infty} (a_k^2 + b_k^2).$$