11.4* Complex Fourier series

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Trigonometric and exponential Fourier series

Let f be a 2π-periodic function,
then its Fourier series is defined by

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big),$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nxdx$$

Complex form of the Fourier series

Instead of trigonometric functions $\cos nx$ and $\sin nx$ we can use complex exponential functions

$$e^{inx} = \cos nx + i \sin nx, \quad e^{-inx} = \cos nx - i \sin nx.$$

Then we also have

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}), \quad \sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}).$$

And the Fourier series can be written as

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Coefficients of complex Fourier series

A straightforward computation gives

$$c_0 = a_0, \quad , c_n = (b_n - ia_n)/2, \; c_{-n} = (b_n + ia_n)/2, \quad n = 1, 2, 3, ...$$

The coefficients could be also found from the orthogonality relation, they are given by the integrals

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-inx} dx.$$

Complex and real series: example

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$$S_f(x) = \frac{1}{3} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi^2 n^2} + \frac{i}{2\pi n} \right) e^{2\pi inx} =$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{\pi^2 n^2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{\pi n}$$

Fourier series: period $2L$

Let f be a $2L$-periodic function,
then its Fourier series is defined by

$$S_f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-\frac{in\pi x}{L}} dx$$

If f is continuous (also at the end-points of the period) then

$$S_f(x) = f(x), \quad \text{otherwise} \quad S_f(x) = \frac{1}{2}\left(f(x+) + f(x-)\right).$$