11.1-11.2. Fourier series

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Fourier series: reminder

Given a 2π-periodic function f, consider the series

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Convergence theorem: reminder

Theorem

Let f be a piece-wise continuous function on $[-\pi, \pi]$ (it can be extended 2π-periodically). Suppose that f has left and right derivatives at each point. Then the Fourier series S_f converges at each point and

$$S_f(x) = \frac{f(x-0) + f(x+0)}{2},$$

where $f(x \pm 0) = \lim_{h \to 0+} f(x \pm h)$.

Further example

Example

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$$a_n = \begin{cases} -\frac{4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} , \quad b_n = 0$$

$$S_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Odd and even functions

f is odd if $f(x) = -f(-x)$.
Fact: f is odd $\Leftrightarrow$ all $a_n = 0$ (cos coefficients)

f is even if $f(x) = f(-x)$.
Fact: f is even $\Leftrightarrow$ all $b_n = 0$ (sin coefficients)

Half range expansions

Given a function f on $(0, \pi)$ we can extend it either as an odd or as even 2π-periodic function expand either as sin or cos Fourier series.

Example: $f(x) = \pi - x$. We have

$$f(x) = a_0 + \sum_1^\infty a_n \cos nx = \sum_1^\infty b_n \sin nx,$$

$$a_0 = \frac{1}{\pi} \int_0^\pi f(x)\, dx,$$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx\, dx, \; n = 1, 2, \ldots$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx\, dx, \; n = 1, 2, \ldots.$$

Period $2L$

Given $f(x)$ such that $f(x) = f(x+2L)$. The corresponding Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx, \;\; n = 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx, \;\; n = 1, 2, \ldots .$$

An old exam problem

The function f is defined by the following conditions:
i) $f(x) = f(-x)$ for all real x.
ii) $f(x) = f(x+4)$ for all real x.
iii) $f(x) = 1 - x$ for $0 < x < 2$.
Sketch the graph of f for $-2 < x < 6$. Find the Fourier series of f.