11.1. Fourier series

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Periodic functions

Function f defined on the *whole* real axis has period p if

$$f(x) = f(x+p) \text{ for all } x \in \mathbf{R}$$

Properties

— If f and g have period p then so does their linear combinations $af(x+p) + bg(x+p) = af(x) + bg(x)$.

— If $f(x+p) = f(x)$ then $f(x+np) = f(x)$ for all integer n
The smallest possible period for a given function is called its fundamental period

— $f(x)$ has period $p \Rightarrow g(x) = f(cx)$ has period p/c.
Therefore it suffices to study the functions of some fixed period. Then we can make scaling. We will deal mainly with functions of period 2π.

— A periodic function with period p can be reconstructed from its values on a segment of length p

Examples

In nature

- Pendulum
- Wave motion
- Crystal structures
- Sound waves.

In mathematics

- $1, \ \sin nx, \ \cos nx \ n = 1, 2, 3, \dots$ - trigonometric system
- e^{inx}, $n = 0, 1, 3, \dots$ - exponential system

They are related by the Euler formulas

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

or

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}, \ \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

Refresh your knowledge on complex numbers !!

Trigonometric and exponential series

Trigonometrical series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Exponential (complex) series:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}.$$

Those are examples of 2π-periodic functions. Our main question: if we know the sum of the series, can we recover the coefficients?

Coefficients of the trigonometric series

Basic formulas:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nxdx$$

Idea of the proof: ORTHOGONALITY.

Orthogonality

Let f and g be two (real-valued) functions defined on an interval $[a, b]$. We say that they are orthogonal on this interval if

$$\int_a^b f(x)g(x)dx = 0$$

For complex-valued function we require

$$\int_a^b f(x)\overline{g(x)}dx = 0$$

The trigonometric and exponential systems are examples of systems with pair-wise orthogonal elements.

Fourier series: idea

Given a 2π-periodic function f, consider the series

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Rectangular wave

Example

Let $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$.

Then

$$S_f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

Convergence theorem

Theorem

Let f be a piece-wise continuous function on $[-\pi, \pi]$ *(it can be extended* 2π*-periodically). Suppose that f has left and right derivatives at each point. Then the Fourier series* S_f *converges at each point and*

$$S_f(x) = \frac{f(x-0) + f(x+0)}{2},$$

where $f(x \pm 0) = \lim_{h \to 0+} f(x \pm h)$.

Further examples

Example

Let $f(x) = |x|, \ -\pi \leq x \leq \pi$ and f is extended to a 2π-periodic function. Find the Fourier series of f.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|dx = \frac{1}{\pi}\int_{0}^{\pi} |x|dx = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nxdx = ... = \left.\frac{2\cos nx}{n^2\pi}\right|_{x=0}^{x=\pi}$$

$$a_n = \begin{cases} -\frac{4}{n^2\pi}, & n \, odd \\ 0, & n \, even \end{cases} \quad , \quad b_n = 0$$

$$S_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$