6.5 (cont.) Applications of the convolution theorem
6.6 Differentiation/integration of the Laplace transform
6.7 Systems of ODE

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Convolution

Let f and g be two piece-wise continuous functions on $[0, +\infty)$. We define a new function

$$h(t) = (f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

It is called the convolution of f and g.
Basic rules for convolutions:

- $f * g = g * f$
- $f * (ag) = a(f * g)$ when a is a constant
- $f * (g_1 + g_2) = f * g_1 + f * g_2$
- $(f * g) * w = f * (g * w)$

Laplace transform: convolution theorem

Theorem

Suppose that f and g are piece-wise continuous functions and there Laplace transforms are defined when $s > a$, $\mathcal{L}\{f\} = F,\ \mathcal{L}\{g\} = G$. Then the Laplace transform of their convolution $f * g$ is also defined when $s > a$ and

$$\mathcal{L}\{f * g\}(s) = F(s)G(s)$$

Convolution: examples

1. $f(t) = t,\ g(t) = t^2,\ f * g = \int_0^t (t-\tau)\tau^2 d\tau = t^4/12$
   Check the convolution theorem
   $\mathcal{L}\{t\} = s^{-2},\ \mathcal{L}\{t^2\} = 2s^{-3}$ and $\mathcal{L}\{t^4/12\} = (24s^{-5}) = 2s^{-5}$.
2. $f(t) = \cos t,\ g(t) = 1,\ f * g = \int_0^t \cos\tau d\tau = \sin t$
   The Laplace transforms are:
   $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$
3. $g(t) = 1,\ f * g = \int_0^t f(\tau)d\tau$ and

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$$

Applications of the convolution theorem, examples

Example

Find the Laplace transform of $f(t) = \int_0^t (t-\tau)^3 \sin(2\tau) d\tau$.

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{t^3\}(s)\mathcal{L}\{\sin 2t\}(s) = \frac{6}{s^4} \cdot \frac{2}{s^2+4} = \frac{12}{s^6+4s^4}$$

Example

Find the inverse Laplace transform of $F(s) = \frac{1}{s^3 - s^2}$

$$\mathcal{L}^{-1}\{(s^3-s^2)^{-1}\} = \mathcal{L}^{-1}\{s^{-2}\} * \mathcal{L}^{-1}\{(s-1)^{-1}\} = t * e^t = \int_0^t e^\tau (t-\tau) d\tau$$

$$(= t(e^t - 1) - (te^t - e^t + 1) = e^t - t - 1)$$

Application to ODE, convolution

Example

Solve the initial value problem

$$y'' + 9y = g(t), \quad y(0) = -1, y'(0) = 9$$

We apply the Laplace transform:

$$Y(s) = \frac{G(s)}{s^2+9} - \frac{s}{s^2+9} + \frac{9}{s^2+9}$$

where G is the Laplace transform of g and then use the convolution theorem

$$y(t) = \frac{1}{3}\int_0^{\tau} g(\tau)\sin 3(t-\tau)d\tau - \cos 3t + 3\sin 3t$$

Compare the answer to the one given by the method of variation of parameters.

Integral equations

Example (exam problem)

Find $y(t)$ that solves the equation

$$y(t) + \int_0^t y(t-\tau)e^{\tau}\,d\tau = \delta(t-5),\ t > 0.$$

We apply the Laplace transform

$$Y(s)(1 + \frac{1}{s-1}) = e^{-5s}, \quad Y(s) = (1 - 1/s)e^{-5s}$$

Then, taking the inverse Laplace transform we get:

$$y(t) = \delta(t-5) - u(t-5)$$

Differentiation and integration of Laplace transforms

Differentiation: $F'(s) = -\mathcal{L}(tf(t))$

Integration: $\int_s^\infty F(\sigma)d\sigma = \mathcal{L}(\frac{1}{t}f(t))$

Example

Find $\mathcal{L}^{-1}\left(\log\left(1+\frac{\omega^2}{s^2}\right)\right)$

$$F(s) = \log\left(1+\frac{\omega^2}{s^2}\right) = \mathcal{L}(f(t)) \;\Rightarrow$$

$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s} = -\mathcal{L}(tf(t)) \;\Rightarrow$$

$$-tf(t) = 2(\cos\omega t - 1) \;\Rightarrow f(t) = \frac{2}{t}(1-\cos\omega t)$$

Systems of ODEs

$$y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + g_1(t)$$
$$y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + g_2(t)$$
$$y_1(0) = y_1,\ y_2(0) = y_2.$$

— Apply the Laplace transform to get a system of equations on $Y_1(s)$, $Y_2(s)$

— solve this system and find $Y_1(s)$, $Y_2(s)$;

— take the inverse Laplace transform

Example

An electrical network (6.7.19), $i_1(0) = i_2(0) = 0$,

$$4i_1 + 8(i_1 - i_2) + 2i_1' = 390\cos t, \quad 8i_2 + 4i_2' + 8(i_2 - i_1) = 0$$

Applying the Laplace transform, we get

$$12I_1 - 8I_2 + 2sI_1 = 390s(s^2+1)^{-1}, \quad 16I_2 - 8I_1 + 4sI_2 = 0$$

Then $2I_1 = (s+4)I_2$ and

$$[(s+6)(s+4) - 8]I_2 = 390s(s^2+1)^{-1}$$

$$I_2 = \frac{390s}{(s^2+1)(s+2)(s+8)}, \, I_1 = \frac{195s(s+4)}{(s^2+1)(s+2)(s+8)}$$

...partial fractions +inverse Laplace

Computations for the last example

From MATLAB `residue`: $\frac{390s}{(s^2+1)(s+2)(s+8)} = \frac{8}{s+8} - \frac{26}{s+2} + \frac{9-6i}{s-i} + \frac{9+6i}{s+i}$

$\frac{390s}{(s^2+1)(s+2)(s+8)} = \frac{8}{s+8} - \frac{26}{s+2} + \frac{18s+12}{s^2+1}$

Then the inverse Laplace transform gives:

$$i_2(t) = 8e^{-8t} - 26e^{-2t} + 18\cos t + 12\sin t$$

Similarly: $\frac{195s^2+780s}{(s^2+1)(s+2)(s+8)} = -\frac{16}{s+8} - \frac{26}{s+2} + \frac{21-7.5i}{s-i} + \frac{21+7.5i}{s+i}$

$\frac{195s^2+780s}{(s^2+1)(s+2)(s+8)} = -\frac{16}{s+8} - \frac{26}{s+2} + \frac{42s+15}{s^2+1}$

$$i_1(t) = -26e^{-8t} - 26e^{-2t} + 42\cos t + 15\sin t$$