**Problem 1** Let $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi, & t \geq \pi \end{cases}$.

**a)** Find the Laplace transform of $f$.

**b)** Solve the initial value problem $y'' + 4y = f(t),\ t \geq 0,\ y(0) = 1,\ y'(0) = 0.$

**Problem 2** Consider the boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0,\ 0 < x < \pi,\ 0 < y < 2\pi, \quad u(0, y) = 0, u_x(\pi, y) = 0 \qquad (*)$$

**a)** Find all solutions of $(*)$ on the form $u(x, y) = F(x)G(y)$.

**b)** Find a solution of $(*)$ that also has the following values on the horizontal sides

$$u(x, 0) = u(x, 2\pi) = \sin\frac{3x}{2} + 4\sin\frac{7x}{2} - 5\sin\frac{11x}{2}.$$

**Problem 3** Find the inverse Fourier transform of the function

$$\frac{1}{(1 + iw)^2}.$$

(Hint: you may use the formula $\mathcal{F}(e^{-x}u(x)) = \frac{1}{\sqrt{2\pi}(1+iw)}$ or you may apply the residue calculus.)

**Problem 4** Let $u(x, y) = e^{2x}\cos by$.

**a)** For which value(s) of $b$ is $u(x, y)$ harmonic?

**b)** Find $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function in the whole complex plane. Justify your answer.

**Problem 5** Let $f(z) = (1 - z)^{-3}$.

**a)** Use the Maclaurin series $(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n$ and term-wise differentiation to find the Maclaurin series of $f(z)$. Find the radius of convergence of this series.

**b)** Write down the Laurent series of the function $f(z)$ with center $z_0 = 0$ that converges in $\{z : |z| > 1\}$.

**Problem 6** Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4x + 5)^2}.$$

# Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero everywhere except $a$ and satisfies $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, moreover $\int_{-\infty}^{\infty} g(t)\delta(t-a) = g(a)$ for any continuous function $g$.
- **Convolution** For functions defined on the real line:
  $f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \ -\infty < x < \infty;$
  for functions defined only on the positive half-axis:
  $f * g(x) = \int_0^x f(y)g(x-y)dy.$

# Laplace transform

- $\mathcal{L}\{f\}(s) = F(s) \int_0^\infty f(t)e^{-st}dt$
- $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
- $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\left\{\int_0^t f(\tau)d\tau)\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$
- $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
- $\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$, $c > 0$
- $\mathcal{L}\{tf(t)\}(s) = -F'(s)$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)d\sigma$

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n, n = 1, 2, ...$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $t^n e^{at}, n = 1, 2, ...$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}$ |
| $\sin bt$ | $\frac{b}{s^2+b^2}$ |
| $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$ |
| $u(t-c), c > 0$ | $\frac{e^{-cs}}{s}$ |
| $\delta(t-c), c > 0$ | $e^{-cs}$ |

## Fourier series and Fourier transform

- Periodic functions with period $2L$, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{l} f(x)\cos\frac{n\pi}{L}xdx, \quad b_n = \frac{1}{L}\int_{-L}^{l} f(x)\sin\frac{n\pi}{L}xdx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

- Parseval's identities $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$, $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$
- $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$
- $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$
- $\hat{f'}(w) = iw\hat{f}(w)$
- $\widehat{f''}(w) = -w^2\hat{f}(w)$
- $\widehat{f(x-a)}(w) = e^{-iaw}\hat{f}(w)$
- $\hat{f}(w-b) = \widehat{e^{ibx}f(x)}(w)$
- $\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$

| $f(x)$ | $\hat{f}(w)$ |
|---|---|
| $\delta(x-a)$ | $\frac{1}{\sqrt{2\pi}}e^{-iaw}$ |
| $\begin{cases}1, & -b \le x \le b\\ 0, & \lvert x\rvert > b\end{cases}$ | $\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$ |
| $e^{-ax}u(x)$ | $\frac{1}{\sqrt{2\pi}(a+iw)}$ |
| $\frac{1}{x^2+a^2}$ | $\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$ |
| $e^{-ax^2}$ | $\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$ |

## Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i\sin y)$,
  $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurant series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \; a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}, \; b_n = \frac{1}{2\pi i}\oint_C f(z)(z-z_0)^{n-1}dz$$