Solutions

Problem 1
a
Let

$$f(t) = \begin{cases} t, \ 0 < t < \pi \\ \pi, \ t \geq \pi \end{cases} .$$

Find the Laplace transform of f.

$$\mathcal{L}f(s) = \int_0^\pi te^{-ts}dt + \int_\pi^\infty e^{-ts}dt = F_1(s) + F_2(s).$$

<u>Solution:</u> Explicit calculation:

$$F_1(s) = -\frac{\pi}{s}e^{-\pi s} - \frac{1}{s^2}e^{-\pi s} + \frac{1}{s^2}; \quad F_2(s) = \frac{\pi}{s}e^{-\pi s}.$$

Finally

$$\underline{\mathcal{L}f(s)\} = \frac{1 - e^{-\pi s}}{s^2}}$$

b
Solve the initial value problem $y'' + 4y = f(t), \ t \geq 0, \ y(0) = 1, \ y'(0) = 0.$
<u>Solution:</u> Let $Y(s) = (\mathcal{L}y)(s)$. We than have

$$y'' + 4y = f(t), \ t \geq 0, \ y(0) = 1, \ y'(0) = 0 \ \Rightarrow$$
$$-s + (s^2+4)Y(s) = \frac{1 - e^{-\pi s}}{s^2} \ \Rightarrow Y(s) = \frac{1 - e^{-\pi s}}{s^2} - \frac{1}{4}\frac{1 - e^{-\pi s}}{s^2+4} + \frac{s}{s^2+4}.$$

Finally

$$y(t) = t - u(t-\pi)(t-\pi) - \frac{1}{8}\sin 2t + \frac{1}{8}u(t-\pi)\sin 2t + \cos 2t.$$

Problem 2 Consider the boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0, \ 0 < x < \pi, \ 0 < y < 2\pi, \quad u(0,y) = 0, u_x(\pi, y) = 0 \qquad (*)$$

a
Find all solutions of $(*)$ on the form $u(x,y) = F(x)G(y)$.
<u>Solution:</u>

$$u(x,y) = F(x)G(y), \ u_{xx} + u_{yy} = 0 \ \Rightarrow \ \begin{cases} F''(x) + kF(x) = 0, \\ G''(y) - kG(y) = 0. \end{cases}$$

$$u(0,y) = 0, u_x(\pi,y) = 0 \ \Rightarrow \ k = (n + \frac{1}{2})^2, \ n = 0, 1, \dots \ .$$

Respectively

$$F_n(x) = \sin(n + \frac{1}{2})x, \quad G_n(y) = A_n e^{(n+1/2)y} + B_n e^{-(n+1/2)y}.$$

b

Find a solution of $(*)$ that also has the following values on the horizontal sides

$$u(x,0) = u(x,2\pi) = \sin\frac{3x}{2} + 4\sin\frac{7x}{2} - 5\sin\frac{11x}{2}.$$

Solution:

$$u(x,y) = \sum_{n=0}^{\infty}[A_n e^{(n+1/2)y} + B_n e^{-(n+1/2)y}]\sin(n+\frac{1}{2})x$$

We have $A_n = B_n = 0$ for $n \neq 1,3,5$.

The rest of the coefficients can be found from the systems

$$\begin{cases} A_1 + B_1 = 1 \\ A_1 e^{3\pi} + B_1 e^{-3\pi} = 1, \end{cases} \quad \begin{cases} A_3 + B_3 = 4 \\ A_3 e^{5\pi} + B_3 e^{-5\pi} = 4, \end{cases} \quad \begin{cases} A_5 + B_5 = -5 \\ A_5 e^{7\pi} + B_5 e^{-7\pi} = -5, \end{cases}$$

Finally

$$A_1 = \frac{1-e^{-3\pi}}{e^{3\pi}-e^{-3\pi}}, \quad B_1 = -\frac{1-e^{3\pi}}{e^{3\pi}-e^{-3\pi}},$$
$$A_3 = 4\frac{1-e^{-5\pi}}{e^{5\pi}-e^{-5\pi}}, \quad B_3 = -4\frac{1-e^{5\pi}}{e^{5\pi}-e^{-5\pi}},$$
$$A_5 = -5\frac{1-e^{-7\pi}}{e^{7\pi}-e^{-7\pi}}, \quad B_5 = 5\frac{1-e^{7\pi}}{e^{7\pi}-e^{-7\pi}},$$

Problem 3 Find the inverse Fourier transform of the function

$$\frac{1}{(1+iw)^2}.$$

(Hint: you may use the formula $\mathcal{F}(e^{-x}u(x)) = \frac{1}{\sqrt{2\pi}(1+iw)}$ or you may apply the residue calculus.)

Solution:

$$\mathcal{F}^{-1}\left(\frac{1}{(1+iw)^2}\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{iwx}}{(1+iw)^2}dw$$

For complex values of w the denominator vanishes at $w = i$. Therefore

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{iwx}}{(1+iw)^2}dw = \begin{cases} 0, & x<0; \\ i\sqrt{2\pi}\mathrm{Res}|_{w=i}\frac{e^{iwx}}{(1+iw)^2} = xe^{-x}, & x>0. \end{cases}$$

Problem 4

a

Let $u(x,y) = e^{2x}\cos by$. For which value(s) of b is $u(x,y)$ harmonic?

Solution: $b = \pm 2$. In this case $u(x,y) = \Re e^{2(x+iy)}$

b

Find $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$ is an analytic function in the whole complex plane. Justify your answer.

Solution: Respectively $f(z) = e^{2z} + ic$, $c \in \mathbb{R}$ and $v(x,y) = e^{2x}\sin 2y + c$.

Problem 5

Let $f(z) = (1-z)^{-3}$.

a

Use the Maclaurin series $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ and term-wise differentiation to find the Maclaurin series of $f(z)$. Find the radius of convergence of this series.

Solution:

$$f(z) = \frac{1}{2}\left(\frac{1}{1-z}\right)'' \Rightarrow f(z) = \sum_{0}^{\infty} n(n+1)z^n.$$

Convergence radius is 1, this is the distance from the centre (at the origin) to the nearest singularity at z=1.

b

Write down the Laurent series of the function $f(z)$ with center $z_0 = 0$ that converges in $\{z : |z| > 1\}$.

Solution:

$$f(z) = -\sum_{0} \infty n(n+1)\frac{1}{z^{n+3}}, \text{ for } |z| > 1.$$

Problem 6

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2}.$$

Solution: Zeros of denominator $z_{\pm} = -2 \pm i$. We have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2} = 2i\pi \text{Res}|_{z=z_+} \frac{1}{(z-z_+)^2(z-z_-)^2} = \frac{\pi}{2}.$$