Problem 1

a) $f(z) = z(z + \bar{z}) = 2xz = 2x^2 + 2ixy$ $u(x, y) = 2x^2$, $v(x, y) = 2xy$.

b) $u_x = 4x$, $v_y = 2x$. $u_x = v_y \;\Rightarrow\; x = 0$.
$v_x = 2y$, $u_y = 0$. $u_y = -v_x \;\Rightarrow\; y = 0$.
The function is analytic at $z = 0$ only. $f'(0) = 0$.

c) u is not a harmonic function. v is a harmonic function.

Problem 2

a) $\overline{f(\bar{z})} = e^{(1-i)z}$.
b) $\overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$.

Problem 3

$a = -3$, $v(x, y) = 3x^2y - y^3 + C$, where C is any constant.

Problem 4

a) $z = (2k + 1)i\pi$, $k = 0, \pm1, \pm2, \ldots$.
b) $z = x + iy \Rightarrow |e^{x+iy}| = e^x \leq e^{|z|}$. Equality takes place if $|z| = x$, i.e. when $y = 0$, $x \geq 0$

Problem 5

The strip $\{w = u + iv : -\infty < u < \infty, -\pi/2 < v < \pi/2\}$

Problem 6

a) $z_1 = e^{i\pi/3}$, $z_2 = -1$, $z_3 = e^{-i\pi/3}$.
$\text{Res}_{z=z_1} f = \frac{1}{3}e^{i\pi/3}(1 + 2e^{i\pi/3})$, $\text{Res}_{z=z_2} f = 1/3$,
$\text{Res}_{z=z_3} f = \frac{1}{3}e^{-i\pi/3}(1 + 2e^{-i\pi/3})$
The integral is zero.

Problem 7

a) $f(z) = \sum_0^\infty (-1)^n z^{2n+1}$
b) $f'(z) = \sum_0^\infty (-1)^n (2n + 1)z^{2n}$, $R = 1$
c) $\frac{z}{z^2+1} = \frac{1}{2}\frac{1}{z+i} + \frac{1}{2}\frac{1}{z-i} = \frac{1}{2}\frac{1}{(z-1)+(1+i)} + \frac{1}{2}\frac{1}{(z-1)+(1-i)}$
For $|z - 1| < \sqrt{2}$ we get $f(z) = \sum_{n=0}^\infty (-1)^n \frac{(1-i)^{n+1}+(1+i)^{n+1}}{2^{n+2}}(z - 1)^n$.
For $|z - 1| > \sqrt{2}$ we get $f(z) = \sum_{n=1}^\infty (-1)^{n-1} \frac{(1+i)^{n-1}+(1-i)^{n-1}}{2}(z - 1)^{-n}$.

Problem 8

a) All isolated singular points are $z_n = 2i\pi n$, $n = 0, \pm1, \pm2, \ldots$. For $n = 0$ we have removable singularity, the rest are simple poles.
b) The integral is zero.

Problem 10

a) $\frac{2\pi}{3\sqrt{3}}$ **b)** $\frac{2\pi}{3}4^{-1/6}$ **c)** $\frac{\pi}{2}e^{-1/\sqrt{2}}\sin(1/\sqrt{2})$