5–8 MAPPING OF CURVES

Find and sketch or graph the images of the given curves under the given mapping.

5. $x = 1, 2, 3, 4, \quad y = 1, 2, 3, 4, \quad w = z^2$
6. **Rotation.** Curves as in Prob. 5, $w = iz$
7. **Reflection in the unit circle.** $|z| = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$, $\text{Arg } z = 0, \pm\pi/4, \pm\pi/2, \pm 3\pi/2$
8. **Translation.** Curves as in Prob. 5, $w = z + 2 + i$
9. CAS **EXPERIMENT. Orthogonal Nets.** Graph the orthogonal net of the two families of level curves $\text{Re} f(z) = \text{const}$ and $\text{Im} f(z) = \text{const}$, where **(a)** $f(z) = z^4$, **(b)** $f(z) = 1/z$, **(c)** $f(z) = 1/z^2$, **(d)** $f(z) = (z + i)/(1 + iz)$. Why do these curves generally intersect at right angles? In your work, experiment to get the best possible graphs. Also do the same for other functions of your own choice. Observe and record shortcomings of your CAS and means to overcome such deficiencies.

10–14 MAPPING OF REGIONS

Sketch or graph the given region and its image under the given mapping.

10. $|z| \leqq \frac{1}{2}, \quad -\pi/8 < \text{Arg } z < \pi/8, \quad w = z^2$
11. $1 < |z| < 3, \quad 0 < \text{Arg } z < \pi/2, \quad w = z^3$
12. $2 \leqq \text{Im } z \leqq 5, \quad w = iz$
13. $x \geqq 1, \quad w = 1/z$
14. $|z - \frac{1}{2}| \leqq \frac{1}{2}, \quad w = 1/z$

15–19 FAILURE OF CONFORMALITY

Find all points at which the mapping is not conformal. Give reason.

15. A cubic polynomial
16. $\dfrac{z + \frac{1}{2}}{4z^2 + 2}$
17. $\sin \pi z$
18. **Magnification of Angles.** Let $f(z)$ be analytic at z_0. Suppose that $f'(z_0) = 0, \cdots, f^{(k-1)}(z_0) = 0$. Then the mapping $w = f(z)$ magnifies angles with vertex at z_0 by a factor k. Illustrate this with examples for $k = 2, 3, 4$.
19. Prove the statement in Prob. 18 for general $k = 1, 2, \cdots$. *Hint.* Use the Taylor series.

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Find the magnification ratio M. Describe what it tells you about the mapping. Where is $M = 1$? Find the Jacobian J.

20. $w = \frac{1}{2}z^2$
21. $w = z^3$
22. $w = 1/z$

17.2 Linear Fractional Transformations (Möbius Transformations)

Conformal mappings can help in modeling and solving boundary value problems by first mapping regions conformally onto another. We shall explain this for standard regions (disks, half-planes, strips) in the next section. For this it is useful to know properties of special basic mappings. Accordingly, let us begin with the following very important class.

The next two sections discuss linear fractional transformations. The reason for our thorough study is that such transformations are useful in modeling and solving boundary value problems, as we shall see in Chapter 18. The task is to get a good grasp of which conformal mappings map certain regions conformally onto each other, such as, say mapping a disk onto a half-plane (Sec. 17.3) and so forth. Indeed, the first step in the modeling process of solving boundary value problems is to identify the correct conformal mapping that is related to the "geometry" of the boundary value problem.

The following class of conformal mappings is very important. **Linear fractional transformations** (or **Möbius transformations**) are mappings

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

where a, b, c, d are complex or real numbers. Differentiation gives

EXAMPLE 8 Estimation of an Integral

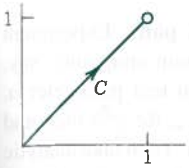


Fig. 344. Path in Example 8

Find an upper bound for the absolute value of the integral

$$\int_C z^2\, dz, \qquad C \text{ the straight-line segment from 0 to } 1 + i, \text{ Fig. 344.}$$

Solution. $L = \sqrt{2}$ and $|f(z)| = |z^2| \leqq 2$ on C gives by (13)

$$\left|\int_C z^2\, dz\right| \leqq 2\sqrt{2} = 2.8284.$$

The absolute value of the integral is $|-\frac{2}{3} + \frac{2}{3}i| = \frac{2}{3}\sqrt{2} = 0.9428$ (see Example 1). ■

Summary on Integration. Line integrals of $f(z)$ can always be evaluated by (10), using a representation (1) of the path of integration. If $f(z)$ is analytic, indefinite integration by (9) as in calculus will be simpler (proof in the next section).

PROBLEM SET 14.1

1–10 FIND THE PATH and sketch it.

1. $z(t) = (1 + \frac{1}{4}i)t, \quad (1 \leqq t \leqq 6)$
2. $z(t) = 3 + i + (1 - i)t, \quad (0 \leqq t \leqq 3)$
3. $z(t) = t + 4t^2 i, \quad (0 \leqq t \leqq 1)$
4. $z(t) = t + (1 - t)^2 i, \quad (-1 \leqq t \leqq 1)$
5. $z(t) = 2 - 2i + \sqrt{5}e^{-it}, \quad (0 \leqq t \leqq 2\pi)$
6. $z(t) = 1 + i + e^{-\pi it}, \quad (0 \leqq t \leqq 2)$
7. $z(t) = + 2e^{\pi it/4}, \quad (0 \leqq t \leqq 2)$
8. $z(t) = 5e^{-it}, \quad (0 \leqq t \leqq \pi/2)$
9. $z(t) = t + i(1 - t)^3, \quad (-2 \leqq t \leqq 2)$
10. $z(t) = 2\cos t + i \sin t, \quad (0 \leqq t \leqq 2\pi)$

11–20 FIND A PARAMETRIC REPRESENTATION and sketch the path.

11. Segment from $(-1, 2)$ to $(1, 4)$
12. From $(0, 0)$ to $(2, 1)$ along the axes
13. Upper half of $|z - 4 + i| = 4$ from $(5, -1)$ to $(-3, -1)$
14. Unit circle, clockwise
15. $4x^2 - y^2 = 4$, the branch through $(0, 2)$
16. Ellipse $4x^2 + 9y^2 = 36$, counterclockwise
17. $|z + a - ib| = r$, clockwise
18. $y = 1/x$ from $(1, 1)$ to $(5, \frac{1}{5})$
19. Parabola $y = 1 - \frac{1}{4}x^2, \quad (-2 \leqq x \leqq 2)$
20. $4(x - 2)^2 + 5(y + 1)^2 = 20$

21–30 INTEGRATION

Integrate by the first method or state why it does not apply and use the second method. Show the details.

21. $\int_C \operatorname{Re} z\, dz$, C the shortest path from $1 + i$ to $5 + 5i$
22. $\int_C \operatorname{Re} z\, dz$, C the parabola $y = 1 + \frac{1}{2}(x - 1)^2$ from $1 + i$ to $3 + 3i$
23. $\int_C e^z\, dz$, C the shortest path from $\pi/2i$ to πi
24. $\int_C \cos 2z\, dz$, C the semicircle $|z| = \pi, x \geqq 0$ from $-\pi i$ to πi
25. $\int_C z \exp(z^2)\, dz$, C from 1 along the axes to i
26. $\int_C (z + z^{-1})\, dz$, C the unit circle, counterclockwise
27. $\int_C \sec^2 z\, dz$, any path from $\pi/4$ to $\pi i/4$
28. $\int_C \left(\frac{5}{z - 2i} - \frac{6}{(z - 2i)^2}\right) dz$, C the circle $|z - 2i| = 4$, clockwise
29. $\int_C \operatorname{Im} z^2\, dz$ counterclockwise around the triangle with vertices $0, 1, i$
30. $\int_C \operatorname{Re} z^2\, dz$ clockwise around the boundary of the square with vertices $0, i, 1 + i, 1$
31. **CAS PROJECT. Integration.** Write programs for the two integration methods. Apply them to problems of your choice. Could you make them into a joint program that also decides which of the two methods to use in a given case?

32. Sense reversal. Verify (5) for $f(z) = z^2$, where C is the segment from $-1 - i$ to $1 + i$.

33. Path partitioning. Verify (6) for $f(z) = 1/z$ and C_1 and C_2 the upper and lower halves of the unit circle.

34. TEAM EXPERIMENT. Integration. (a) Comparison. First write a short report comparing the essential points of the two integration methods.

(b) Comparison. Evaluate $\int_C f(z)\,dz$ by Theorem 1 and check the result by Theorem 2, where:

(i) $f(z) = z^4$ and C is the semicircle $|z| = 2$ from $-2i$ to $2i$ in the right half-plane,

(ii) $f(z) = e^{2z}$ and C is the shortest path from 0 to $1 + 2i$.

(c) Continuous deformation of path. Experiment with a family of paths with common endpoints, say, $z(t) = t + ia \sin t$, $0 \leqq t \leqq \pi$, with real parameter a. Integrate nonanalytic functions (Re z, Re (z^2), etc.) and explore how the result depends on a. Then take analytic functions of your choice. (Show the details of your work.) Compare and comment.

(d) Continuous deformation of path. Choose another family, for example, semi-ellipses $z(t) = a \cos t + i \sin t$, $-\pi/2 \leqq t \leqq \pi/2$, and experiment as in (c).

35. *ML*-inequality. Find an upper bound of the absolute value of the integral in Prob. 21.

14.2 Cauchy's Integral Theorem

This section is the focal point of the chapter. We have just seen in Sec. 14.1 that a line integral of a function $f(z)$ generally depends not merely on the endpoints of the path, but also on the choice of the path itself. This dependence often complicates situations. Hence conditions under which this does ***not*** occur are of considerable importance. Namely, if $f(z)$ is analytic in a domain D and D is simply connected (see Sec. 14.1 and also below), then the integral will not depend on the choice of a path between given points. This result (Theorem 2) follows from Cauchy's integral theorem, along with other basic consequences that make ***Cauchy's integral theorem the most important theorem in this chapter*** and fundamental throughout complex analysis.

Let us continue our discussion of simple connectedness which we started in Sec. 14.1.

1. A **simple closed path** is a closed path (defined in Sec. 14.1) that does not intersect or touch itself as shown in Fig. 345. For example, a circle is simple, but a curve shaped like an 8 is not simple.

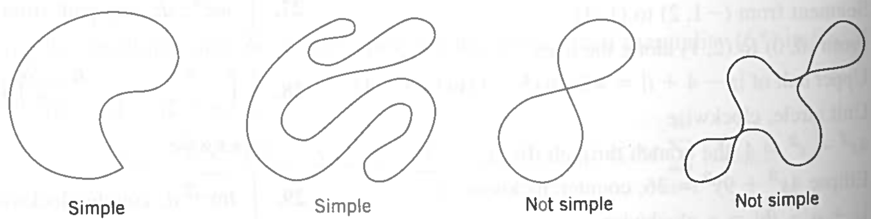


Fig. 345. Closed paths

2. A **simply connected domain** D in the complex plane is a domain (Sec. 13.3) such that every simple closed path in D encloses only points of D. *Examples:* The interior of a circle ("open disk"), ellipse, or any simple closed curve. A domain that is not simply connected is called **multiply connected.** *Examples:* An annulus (Sec. 13.3), a disk without the center, for example, $0 < |z| < 1$. See also Fig. 346.

More precisely, a **bounded domain** D (that is, a domain that lies entirely in some circle about the origin) is called ***p*-fold connected** if its boundary consists of p closed

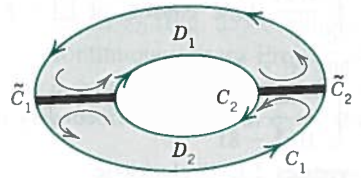


Fig. 354. Doubly connected domain

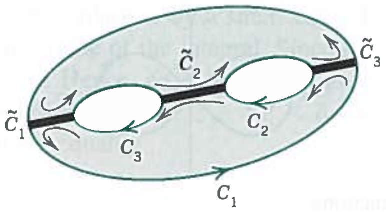


Fig. 355. Triply connected domain

PROBLEM SET 14.2

1–8 COMMENTS ON TEXT AND EXAMPLES

1. Cauchy's Integral Theorem. Verify Theorem 1 for the integral of z^2 over the boundary of the square with vertices $\pm 1 \pm i$. *Hint.* Use deformation.

2. For what contours C will it follow from Theorem 1 that

(a) $\displaystyle\int_C \frac{dz}{z-1} = 0$, (b) $\displaystyle\int_C \frac{\exp(1/z^2)}{z^2+4} = 0$

3. Deformation principle. Can we conclude from Example 4 that the integral is also zero over the contour in Prob. 1?

4. If the integral of a function over the unit circle equals 2 and over the circle of radius 3 equals 6, can the function be analytic everywhere in the annulus $1 < |z| < 3$?

5. Connectedness. What is the connectedness of the domain in which $(\cos z^2)/(z^4 + 1)$ is analytic?

6. Path independence. Verify Theorem 2 for the integral of e^z from 0 to $1 + i$ (a) over the shortest path and (b) over the x-axis to 1 and then straight up to $1 + i$.

7. Deformation. Can we conclude in Example 2 that the integral of $1/(z^2 + 4)$ over (a) $|z - 2| = 2$ and (b) $|z - 2| = 3$ is zero?

8. TEAM EXPERIMENT. Cauchy's Integral Theorem.

(a) Main Aspects. Each of the problems in Examples 1–5 explains a basic fact in connection with Cauchy's theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of those facts.

(b) Partial fractions. Write $f(z)$ in terms of partial fractions and integrate it counterclockwise over the unit circle, where

(i) $f(z) = \dfrac{2z + 3i}{z^2 + \frac{1}{4}}$ (ii) $f(z) = \dfrac{z+1}{z^2 + 2z}$.

(c) Deformation of path. Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, $z(t) = t + ia(t - t^2)$, $0 \leqq t \leqq 1$, a a real constant, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., z, $\text{Im } z$, z^2, $\text{Re } z^2$, $\text{Im } z^2$, etc.).

9–19 CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

9. $f(z) = \exp(z^2)$
10. $f(z) = \tan \frac{1}{4} z$
11. $f(z) = 1/(4z - 1)$
12. $f(z) = \bar{z}^3$
13. $f(z) = 1/(z^4 - 1.2)$
14. $f(z) = 1/\bar{z}$
15. $f(z) = \text{Re } z$
16. $f(z) = 1/(\pi z - 1)$
17. $f(z) = 1/|z|^2$
18. $f(z) = 1/(5z - 1)$
19. $f(z) = z^3 \cot z$

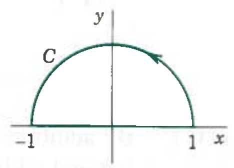
20–30 FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

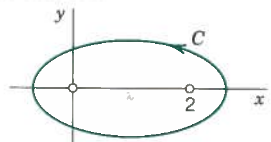
20. $\displaystyle\oint_C \text{Ln}(1 - z)\, dz$, C the boundary of the parallelogram with vertices $\pm i, \pm(1 + i)$.

21. $\displaystyle\oint_C \frac{dz}{z - 2i}$, C the circle $|z| = \pi$ counterclockwise.

22. $\displaystyle\oint_C \text{Re } z\, dz$, C:

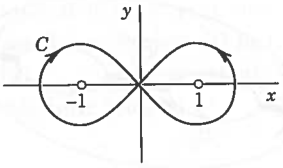


23. $\displaystyle\oint_C \frac{2z - 1}{z^2 - z}\, dz$, C:



Use partial fractions.

24. $\oint_C \frac{dz}{z^2 - 1}$, C:



Use partial fractions.

25. $\oint_C \frac{e^z}{z}\, dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

26. $\oint_C \coth \frac{1}{2}z\, dz$, C the circle $|z - \frac{1}{2}\pi i| = 1$ clockwise.

27. $\oint_C \frac{\cos z}{z}\, dz$, C consists of $|z| = 1$ counterclockwise and $|z| = 3$ clockwise.

28. $\oint_C \frac{\tan \frac{1}{2}z}{16z^4 - 81}\, dz$, C the boundary of the square with vertices $\pm 1, \pm i$ clockwise.

29. $\oint_C \frac{\sin z}{z + 4iz}\, dz$, C: $|z - 4 - 2i| = 6.5$.

30. $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2}\, dz$, C: $|z - 2| = 4$ clockwise. Use partial fractions.

14.3 Cauchy's Integral Formula

Cauchy's integral theorem leads to Cauchy's integral formula. This formula is useful for evaluating integrals as shown in this section. It has other important roles, such as in proving the surprising fact that analytic functions have derivatives of all orders, as shown in the next section, and in showing that all analytic functions have a Taylor series representation (to be seen in Sec. 15.4).

THEOREM 1 **Cauchy's Integral Formula**

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 356),

(1) $$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) \qquad \textbf{(Cauchy's integral formula)}$$

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

(1*) $$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \qquad \textbf{(Cauchy's integral formula).}$$

PROOF By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. Inserting this into (1) on the left and taking the constant factor $f(z_0)$ out from under the integral sign, we have

(2) $$\oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0) \oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz.$$

The first term on the right equals $f(z_0) \cdot 2\pi i$, which follows from Example 6 in Sec. 14.2 with $m = -1$. If we can show that the second integral on the right is zero, then it would prove the theorem. Indeed, we can. The integrand of the second integral is analytic, except