Type	New Variables		Normal Form
Hyperbolic	$v = \Phi$	$w = \Psi$	$u_{vw} = F_1$
Parabolic	$v = x$	$w = \Phi = \Psi$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \Psi)$	$w = \frac{1}{2i}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

Here, $\Phi = \Phi(x, y), \Psi = \Psi(x, y), F_1 = F_1(v, w, u, u_v, u_w)$, etc., and we denote u as function of v, w again by u, for simplicity. We see that the normal form of a hyperbolic PDE is as in d'Alembert's solution. In the parabolic case we get just one family of solutions $\Phi = \Psi$. In the elliptic case, $i = \sqrt{-1}$, and the characteristics are complex and are of minor interest. For derivation, see Ref. [GenRef3] in App. 1.

EXAMPLE 1 D'Alembert's Solution Obtained Systematically

The theory of characteristics gives d'Alembert's solution in a systematic fashion. To see this, we write the wave equation $u_{tt} - c^2u_{xx} = 0$ in the form (14) by setting $y = ct$. By the chain rule, $u_t = u_y y_t = cu_y$ and $u_{tt} = c^2u_{yy}$. Division by c^2 gives $u_{xx} - u_{yy} = 0$, as stated before. Hence the characteristic equation is $y'^2 - 1 = (y' + 1)(y' - 1) = 0$. The two families of solutions (characteristics) are $\Phi(x, y) = y + x = \text{const}$ and $\Psi(x, y) = y - x = \text{const}$. This gives the new variables $v = \Phi = y + x = ct + x$ and $w = \Psi = y - x = ct - x$ and d'Alembert's solution $u = f_1(x + ct) + f_2(x - ct)$. ■

PROBLEM SET 12.4

1. Show that c is the speed of each of the two waves given by (4).
2. Show that, because of the boundary conditions (2), Sec. 12.3, the function f in (13) of this section must be odd and of period $2L$.
3. If a steel wire 2 m in length weighs 0.9 nt (about 0.20 lb) and is stretched by a tensile force of 300 nt (about 67.4 lb), what is the corresponding speed of transverse waves?
4. What are the frequencies of the eigenfunctions in Prob. 3?

5–8 GRAPHING SOLUTIONS

Using (13) sketch or graph a figure (similar to Fig. 291 in Sec. 12.3) of the deflection $u(x, t)$ of a vibrating string (length $L = 1$, ends fixed, $c = 1$) starting with initial velocity 0 and initial deflection (k small, say, $k = 0.01$).

5. $f(x) = k \sin \pi x$
6. $f(x) = k(1 - \cos \pi x)$
7. $f(x) = k \sin 2\pi x$
8. $f(x) = kx(1 - x)$

9–18 NORMAL FORMS

Find the type, transform to normal form, and solve. Show your work in detail.

9. $u_{xx} + 4u_{yy} = 0$
10. $u_{xx} - 16u_{yy} = 0$
11. $u_{xx} + 2u_{xy} + u_{yy} = 0$
12. $u_{xx} - 2u_{xy} + u_{yy} = 0$
13. $u_{xx} + 5u_{xy} + 4u_{yy} = 0$
14. $xu_{xy} - yu_{yy} = 0$
15. $xu_{xx} - yu_{xy} = 0$
16. $u_{xx} + 2u_{xy} + 10u_{yy} = 0$
17. $u_{xx} - 4u_{xy} + 5u_{yy} = 0$
18. $u_{xx} - 6u_{xy} + 9u_{yy} = 0$
19. **Longitudinal Vibrations of an Elastic Bar or Rod.** These vibrations in the direction of the x-axis are modeled by the wave equation $u_{tt} = c^2u_{xx}$, $c^2 = E/\rho$ (see Tolstov [C9], p. 275). If the rod is fastened at one end, $x = 0$, and free at the other, $x = L$, we have $u(0, t) = 0$ and $u_x(L, t) = 0$. Show that the motion corresponding to initial displacement $u(x, 0) = f(x)$ and initial velocity zero is

$$u = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n ct,$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin p_n x \, dx, \qquad p_n = \frac{(2n + 1)\pi}{2L}.$$

20. **Tricomi and Airy equations.**[2] Show that the *Tricomi equation* $yu_{xx} + u_{yy} = 0$ is of mixed type. Obtain the **Airy equation** $G'' - yG = 0$ from the Tricomi equation by separation. (For solutions, see p. 446 of Ref. [GenRef1] listed in App. 1.)

[2]Sir GEORGE BIDELL AIRY (1801–1892), English mathematician, known for his work in elasticity. FRANCESCO TRICOMI (1897–1978), Italian mathematician, who worked in integral equations and functional analysis.

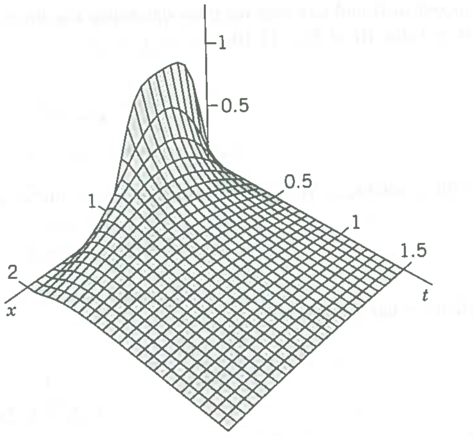

Fig. 300. Solution (20) in Example 4

PROBLEM SET 12.7

1. CAS PROJECT. Heat Flow. (a) Graph the basic Fig. 299.

(b) In (a) apply animation to "see" the heat flow in terms of the decrease of temperature.

(c) Graph $u(x, t)$ with $c = 1$ as a surface over a rectangle of the form $-a < x < a, \quad 0 < y < b$.

2–8 SOLUTION IN INTEGRAL FORM

Using (6), obtain the solution of (1) in integral form satisfying the initial condition $u(x, 0) = f(x)$, where

2. $f(x) = 1$ if $|x| < a$ and 0 otherwise

3. $f(x) = 1/(1 + x^2)$.
Hint. Use (15) in Sec. 11.7.

4. $f(x) = e^{-|x|}$

5. $f(x) = |x|$ if $|x| < 1$ and 0 otherwise

6. $f(x) = x$ if $|x| < 1$ and 0 otherwise

7. $f(x) = (\sin x)/x$.
Hint. Use Prob. 4 in Sec. 11.7.

8. Verify that u in the solution of Prob. 7 satisfies the initial condition.

9–12 CAS PROJECT. Error Function.

$$\text{(21)} \qquad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2}\, dw$$

This function is important in applied mathematics and physics (probability theory and statistics, thermodynamics, etc.) and fits our present discussion. Regarding it as a typical case of a special function defined by an integral that cannot be evaluated as in elementary calculus, do the following.

9. Graph the **bell-shaped curve** [the curve of the integrand in (21)]. Show that erf x is odd. Show that

$$\int_a^b e^{-w^2}\, dw = \frac{\sqrt{\pi}}{2}(\operatorname{erf} b - \operatorname{erf} a).$$

$$\int_{-b}^b e^{-w^2}\, dw = \sqrt{\pi} \operatorname{erf} b.$$

10. Obtain the Maclaurin series of erf x from that of the integrand. Use that series to compute a table of erf x for $x = 0(0.01)3$ (meaning $x = 0, 0.01, 0.02, \cdots, 3$).

11. Obtain the values required in Prob. 10 by an integration command of your CAS. Compare accuracy.

12. It can be shown that erf $(\infty) = 1$. Confirm this experimentally by computing erf x for large x.

13. Let $f(x) = 1$ when $x > 0$ and 0 when $x < 0$. Using erf $(\infty) = 1$, show that (12) then gives

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2c\sqrt{t})}^{\infty} e^{-z^2} dz$$

$$= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(-\frac{x}{2c\sqrt{t}}\right) \qquad (t > 0).$$

14. Express the temperature (13) in terms of the error function.

15. Show that $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-s^2/2}\, ds$

$$= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

Here, the integral is the definition of the "distribution function of the normal probability distribution" to be discussed in Sec. 24.8.

EXAMPLE 5 Polynomials, Rational Functions

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function**. This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–8 REGIONS OF PRACTICAL INTEREST

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 2i| \leqq \frac{1}{4}$
2. $0 < |z| < 1$
3. $\frac{\pi}{2} < |z - 1 + 2i| < \pi$
4. $-\pi < \text{Im } z < \pi$
5. $|\arg z| < \frac{\pi}{3}$
6. $\text{Re}(1/z) < 1$
7. $\text{Re } z \leqq -1$
8. $|z + i| \geqq |z - i|$
9. **WRITING PROJECT. Sets in the Complex Plane.** Write a report by formulating the corresponding portions of the text in your own words and illustrating them with examples of your own.

COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 Function Values. Find Re f, and Im f and their values at the given point z.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$
11. $f(z) = 1/(1 + z)$ at $1 - i$
12. $f(z) = (z - 1)/(z + 1)$ at $2i$
13. **CAS PROJECT. Graphing Functions.** Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves Re $f(z)$ = const and Im $f(z)$ = const in the same figure, and the curves $|f(z)|$ = const in another figure, where **(a)** $f(z) = z^2$, **(b)** $f(z) = 1/z$, **(c)** $f(z) = z^4$.

14–17 Continuity. Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\text{Re } z^2)/|z|$
15. $|z|^2 \text{ Im}(1/z)$
16. $(\text{Im } z^2)/|z|^2$
17. $(\text{Re } z)/(1 - |z|)$

18–23 Differentiation. Find the value of the derivative of

18. $(z - i)/(z + i)$ at i
19. $(z - 2i)^3$ at $= 5 + 2i$
20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.
21. $i(1 - z)^n$ at 0
22. $(iz^3 + 3z^2)^3$ at $2i$
23. $z^3/(z - i)^3$ at $-i$
24. **TEAM PROJECT. Limit, Continuity, Derivative**

    (a) Limit. Prove that (1) is equivalent to the pair of relations

    $$\lim_{z \to z_0} \text{Re } f(z) = \text{Re } l, \qquad \lim_{z \to z_0} \text{Im } f(z) = \text{Im } l.$$

    (b) Limit. If $\lim_{z \to z_0} f(x)$ exists, show that this limit is unique.

    (c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.