

The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–13 proceeds as for ODEs. Problems 16–23 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

EXAMPLE 2 Solving $u_{xx} - u = 0$ Like an ODE

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y .

Solution. Since no y -derivatives occur, we can solve this PDE like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^x + Be^{-x}$ with constant A and B . Here A and B may be functions of y , so that the answer is

$$u(x, y) = A(y)e^x + B(y)e^{-x}$$

with arbitrary functions A and B . We thus have a great variety of solutions. Check the result by differentiation. ■

EXAMPLE 3 Solving $u_{xy} = -u_x$ Like an ODE

Find solutions $u = u(x, y)$ of this PDE.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln |p| = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x ,

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x) dx,$$

here, $f(x)$ and $g(y)$ are arbitrary. ■

PROBLEM SET 12.1

1. **Fundamental theorem.** Prove it for second-order PDEs in two and three independent variables. *Hint.* Prove it by substitution.

2–13 VERIFICATION OF SOLUTIONS

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

2–5 Wave Equation (1) with suitable c

2. $u = x^2 + t^2$
3. $u = \cos 4t \sin 2x$
4. $u = \sin kct \cos kx$
5. $u = \sin at \sin bx$

6–9 Heat Equation (2) with suitable c

6. $u = e^{-t} \sin x$
7. $u = e^{-\omega^2 c^2 t} \cos \omega x$
8. $u = e^{-9t} \sin \omega x$
9. $u = e^{-\pi^2 t} \cos 25x$

10–13 Laplace Equation (3)

10. $u = e^x \cos y, e^x \sin y$
11. $u = \arctan (y/x)$
12. $u = \cos y \sinh x, \sin y \cosh x$

13. $u = x/(x^2 + y^2), y/(x^2 + y^2)$

14. TEAM PROJECT. Verification of Solutions

(a) **Wave equation.** Verify that $u(x, t) = v(x + ct) + w(x - ct)$ with any twice differentiable functions v and w satisfies (1).

(b) **Poisson equation.** Verify that each u satisfies (4) with $f(x, y)$ as indicated.

$$\begin{array}{ll} u = y/x & f = 2y/x^3 \\ u = \sin xy & f = (x^2 + y^2) \sin xy \\ u = e^{x^2 - y^2} & f = 4(x^2 + y^2)e^{x^2 - y^2} \\ u = 1/\sqrt{x^2 + y^2} & f = (x^2 + y^2)^{-3/2} \end{array}$$

(c) **Laplace equation.** Verify that

$u = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies (6) and $u = \ln(x^2 + y^2)$ satisfies (3). Is $u = 1/\sqrt{x^2 + y^2}$ a solution of (3)? Of what Poisson equation?

(d) Verify that u with any (sufficiently often differentiable) v and w satisfies the given PDE.

$$\begin{array}{ll} u = v(x) + w(y) & u_{xy} = 0 \\ u = v(x)w(y) & uu_{xy} = u_x u_y \\ u = v(x + 2t) + w(x - 2t) & u_{tt} = 4u_{xx} \end{array}$$

15. **Boundary value problem.** Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation

(3) and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

16–23 PDEs SOLVABLE AS ODEs

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for $u = u(x, y)$:

16. $u_{yy} = 0$

17. $u_{xx} + 16\pi^2 u = 0$

18. $25u_{yy} - 4u = 0$

19. $u_y + y^2 u = 0$

20. $2u_{xx} + 9u_x + 4u = -3 \cos x - 29 \sin x$

21. $u_{yy} + 6u_y + 13u = 4e^{3y}$

22. $u_{xy} = u_x$

23. $x^2 u_{xx} + 2x u_x - 2u = 0$

24. **Surface of revolution.** Show that the solutions $z = z(x, y)$ of $yz_x = xz_y$ represent surfaces of revolution. Give examples. *Hint.* Use polar coordinates r, θ and show that the equation becomes $z_\theta = 0$.

25. **System of PDEs.** Solve $u_{xx} = 0, u_{yy} = 0$

12.2 Modeling: Vibrating String, Wave Equation

In this section we model a vibrating string, which will lead to our first important PDE, that is, equation (3) which will then be solved in Sec. 12.3. *The student should pay very close attention to this delicate modeling process and detailed derivation starting from scratch*, as the skills learned can be applied to modeling other phenomena in general and in particular to modeling a vibrating membrane (Sec. 12.7).

We want to derive the PDE modeling small transverse vibrations of an elastic string, such as a violin string. We place the string along the x -axis, stretch it to length L , and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 286.

$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived. This PDE should not be too complicated, so that we can solve it. Reasonable simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions

1. The mass of the string per unit length is constant (“homogeneous string”). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

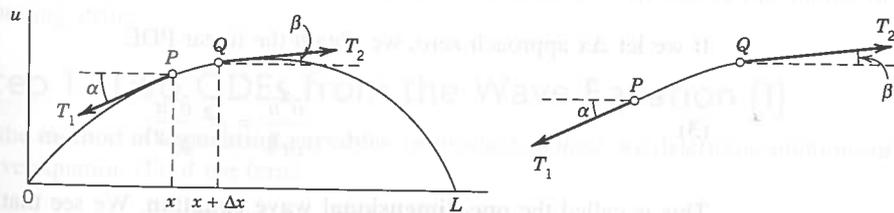
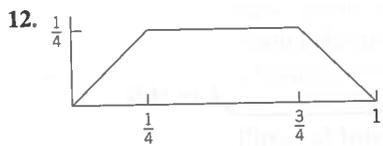
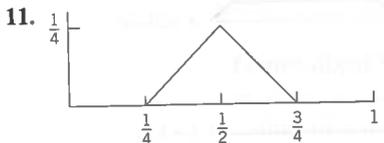
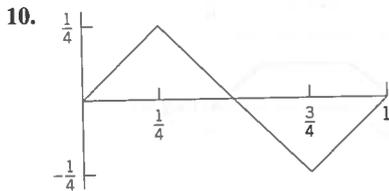
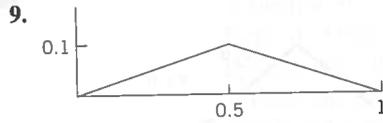


Fig. 286. Deflected string at fixed time t . Explanation on p. 544

7. $kx(1-x)$

8. $kx^2(1-x)$



13. $2x - 4x^2$ if $0 < x < \frac{1}{2}$, 0 if $\frac{1}{2} < x < 1$

14. **Nonzero initial velocity.** Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01x$ if $0 \leq x \leq \frac{1}{2}\pi$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{2}\pi \leq x \leq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)

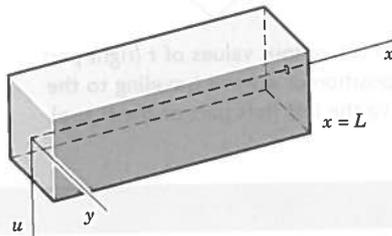


Fig. 292. Elastic beam

15-20 SEPARATION OF A FOURTH-ORDER PDE. VIBRATING BEAM

By the principles used in modeling the string it can be shown that small free vertical vibrations of a uniform elastic beam (Fig. 292) are modeled by the fourth-order PDE

(21)
$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \quad (\text{Ref. [C11]})$$

where $c^2 = EI/\rho A$ (E = Young's modulus of elasticity, I = moment of inertia of the cross section with respect to the

y -axis in the figure, ρ = density, A = cross-sectional area). (Bending of a beam under a load is discussed in Sec. 3.3.)

15. Substituting $u = F(x)G(t)$ into (21), show that

$$F^{(4)}/F = -\ddot{G}/c^2 G = \beta^4 = \text{const},$$

$$F(x) = A \cos \beta x + B \sin \beta x$$

$$+ C \cosh \beta x + D \sinh \beta x,$$

$$G(t) = a \cos c\beta^2 t + b \sin c\beta^2 t.$$

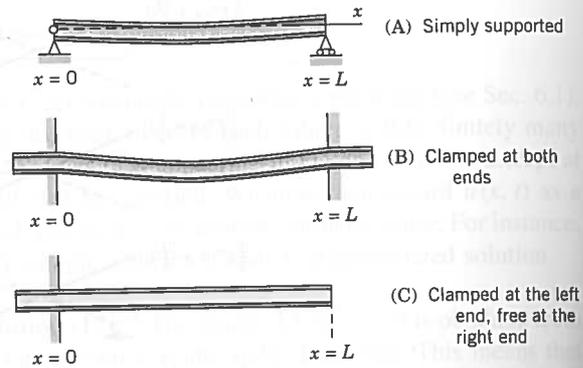


Fig. 293. Supports of a beam

16. **Simply supported beam in Fig. 293A.** Find solutions $u_n = F_n(x)G_n(t)$ of (21) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 293A)

$$u(0, t) = 0, u(L, t) = 0$$

(ends simply supported for all times t),

$$u_{xx}(0, t) = 0, u_{xx}(L, t) = 0$$

(zero moments, hence zero curvature, at the ends).

17. Find the solution of (21) that satisfies the conditions in Prob. 16 as well as the initial condition

$$u(x, 0) = f(x) = x(L-x).$$

18. Compare the results of Probs. 17 and 7. What is the basic difference between the frequencies of the normal modes of the vibrating string and the vibrating beam?

19. **Clamped beam in Fig. 293B.** What are the boundary conditions for the clamped beam in Fig. 293B? Show that F in Prob. 15 satisfies these conditions if βL is a solution of the equation

(22)
$$\cosh \beta L \cos \beta L = 1.$$

Determine approximate solutions of (22), for instance, graphically from the intersections of the curves of $\cos \beta L$ and $1/\cosh \beta L$.

conditions. State the difference between Fig. 291 in Sec. 12.3 and Fig. 295.

5-7 **LATERALLY INSULATED BAR**

Find the temperature $u(x, t)$ in a bar of silver of length 10 cm and constant cross section of area 1 cm^2 (density 10.6 g/cm^3 , thermal conductivity $1.04 \text{ cal/(cm sec } ^\circ\text{C)}$, specific heat $0.056 \text{ cal/(g } ^\circ\text{C)}$) that is perfectly insulated laterally, with ends kept at temperature 0°C and initial temperature $f(x)^\circ\text{C}$, where

- 5. $f(x) = \sin 0.1\pi x$
- 6. $f(x) = 4 - 0.8|x - 5|$
- 7. $f(x) = x(10 - x)$

8. **Arbitrary temperatures at ends.** If the ends $x = 0$ and $x = L$ of the bar in the text are kept at constant temperatures U_1 and U_2 , respectively, what is the temperature $u_1(x)$ in the bar after a long time (theoretically, as $t \rightarrow \infty$)? First guess, then calculate.

9. In Prob. 8 find the temperature at any time.

10. **Change of end temperatures.** Assume that the ends of the bar in Probs. 5-7 have been kept at 100°C for a long time. Then at some instant, call it $t = 0$, the temperature at $x = L$ is suddenly changed to 0°C and kept at 0°C , whereas the temperature at $x = 0$ is kept at 100°C . Find the temperature in the middle of the bar at $t = 1, 2, 3, 10, 50$ sec. First guess, then calculate.

BAR UNDER ADIABATIC CONDITIONS

“Adiabatic” means no heat exchange with the neighborhood, because the bar is completely insulated, also at the ends. *Physical Information:* The heat flux at the ends is proportional to the value of $\partial u/\partial x$ there.

11. Show that for the completely insulated bar, $u_x(0, t) = 0$, $u_x(L, t) = 0$, $u(x, t) = f(x)$ and separation of variables gives the following solution, with A_n given by (2) in Sec. 11.3.

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(cn\pi/L)^2 t}$$

12-15 Find the temperature in Prob. 11 with $L = \pi$, $c = 1$, and

- 12. $f(x) = x$
- 13. $f(x) = 1$
- 14. $f(x) = \cos 2x$
- 15. $f(x) = 1 - x/\pi$

16. **A bar with heat generation** of constant rate $H (> 0)$ is modeled by $u_t = c^2 u_{xx} + H$. Solve this problem if $L = \pi$ and the ends of the bar are kept at 0°C . *Hint.* Set $u = v - Hx(x - \pi)/(2c^2)$.

17. **Heat flux.** The *heat flux* of a solution $u(x, t)$ across $x = 0$ is defined by $\phi(t) = -Ku_x(0, t)$. Find $\phi(t)$ for the solution (9). Explain the name. Is it physically understandable that ϕ goes to 0 as $t \rightarrow \infty$?

18-25 **TWO-DIMENSIONAL PROBLEMS**

18. **Laplace equation.** Find the potential in the rectangle $0 \leq x \leq 20, 0 \leq y \leq 40$ whose upper side is kept at potential 110 V and whose other sides are grounded.

19. Find the potential in the square $0 \leq x \leq 2, 0 \leq y \leq 2$ if the upper side is kept at the potential $1000 \sin \frac{1}{2}\pi x$ and the other sides are grounded.

20. **CAS PROJECT. Isotherms.** Find the steady-state solutions (temperatures) in the square plate in Fig. 297 with $a = 2$ satisfying the following boundary conditions. Graph isotherms.

- (a) $u = 80 \sin \pi x$ on the upper side, 0 on the others.
- (b) $u = 0$ on the vertical sides, assuming that the other sides are perfectly insulated.
- (c) Boundary conditions of your choice (such that the solution is not identically zero).

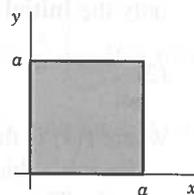


Fig. 297. Square plate

21. **Heat flow in a plate.** The faces of the thin square plate in Fig. 297 with side $a = 24$ are perfectly insulated. The upper side is kept at 25°C and the other sides are kept at 0°C . Find the steady-state temperature $u(x, y)$ in the plate.

22. Find the steady-state temperature in the plate in Prob. 21 if the lower side is kept at $U_0^\circ\text{C}$, the upper side at $U_1^\circ\text{C}$, and the other sides are kept at 0°C . *Hint:* Split into two problems in which the boundary temperature is 0 on three sides for each problem.

23. **Mixed boundary value problem.** Find the steady-state temperature in the plate in Prob. 21 with the upper and lower sides perfectly insulated, the left side kept at 0°C , and the right side kept at $f(y)^\circ\text{C}$.

24. **Radiation.** Find steady-state temperatures in the rectangle in Fig. 296 with the upper and left sides perfectly insulated and the right side radiating into a medium at 0°C according to $u_x(a, y) + hu(a, y) = 0$, $h > 0$ constant. (You will get many solutions since no condition on the lower side is given.)

25. Find formulas similar to (17), (18) for the temperature in the rectangle R of the text when the lower side of R is kept at temperature $f(x)$ and the other sides are kept at 0°C .